Legendrian currents, support functions and tensor-valued measures on singular spaces

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Contents

Abstract

In the study of geometric and topological properties of subanalytic sets (or more generally of constructible functions), the normal cycle is an extremely useful tool. The normal cycle of a subanalytic set is a closed current on the unit tangent bundle of the ambient space which encodes the geometry of the Gauss mapping of the set. Up to now, only complicated constructions, based on Geometric Measure Theory, Sheaf Theory or Stratified Morse Theory were known.
In this work an elementary construction of the normal cycle is presented, which only uses simple properties of subanalytic sets, for instance cell-decompositions.
The essential notion is that of a support function. Support functions of subanalytic sets were introduced by Ludwig Bröcker. A first result of this work is that a constructible function has compact support if and only if its support function is Lipschitz.
It is then shown that subanalytic Legendrian cycles also admit Lipschitz continuous support functions. The normal cycle of a constructible function is characterized by the fact that both support functions coincide.
The main result of this work is an existence and uniqueness result for the normal cycle of a compactly supported constructible function.
As an application of the normal cycle construction, we introduce a sequence of tensor-valued measures. Some of them generalize classical curvature tensors of Riemannian manifolds (like scalar curvature, Einstein tensor, Riemann tensor).

Introduction

The normal cycle construction is a convenient way to define curvature measures of certain *singular spaces*, like convex bodies, sets with positive reach, or subanalytic sets. To each such set, one can associate in a canonical way a closed Federer-Fleming current on the unit tangent bundle of the ambient space which encodes the geometry of the Gauss map. For submanifolds, the normal cycle is just the unit normal bundle.

While in the case of convex sets or sets with positive reach the normal cycle is very easy to describe, its construction in the subanalytic category is much more involved. Using deep tools from Sheaf Theory, Kashiwara-Shapira constructed the normal cycle (under the name *characteristic cycle*, [**33**]). A construction based on Geometric Measure Theory was found by J. Fu ([**21**]–[**26**]). In his construction, the normal cycle of a compact subanalytic set $X \subset \mathbb{R}^n$ is the limit of a sequence of integral cycles associated to so-called *subanalytic auras* of X. The existence of the limit is a consequence of the Federer-Fleming compactness theorem. One of the difficulties is to show that this limit does not depend on the choice of a subanalytic aura. This is achieved by Fu's uniqueness theorem, a version of which we will prove in Chapters I–III. The normal cycle of X is a compactly supported subanalytic cycle on $\mathbb{R}^n \times S^{n-1}$ which vanishes on the contact form, i.e. a Legendrian cycle.

Trying to understand the ideal structure of the ring of constructible functions under Euler multiplication (or convolution), L. Bröcker defined the support function of a (non necessarily compact) subanalytic set and more generally of a constructible function. This is a function on $\mathbb{R}^n$ with values in the group ring $\mathbb{Z}[\mathbb{R}]$. Bröcker showed that X can be recovered from its support transform and characterized the image of the support transform as the set of homogeneous, subanalytic functions $\mathbb{R}^n \to \mathbb{Z}[\mathbb{R}]$.

It turns out that, using Slicing Theory, one can also associate to any compact Legendrian integral cycle an almost everywhere defined support function on $\mathbb{R}^n$ with values in $\mathbb{Z}[\mathbb{R}]$. The normal cycle of a compact, subanalytic set $X \subset \mathbb{R}^n$ is then characterized by the fact that its support function is almost everywhere equal to the support function of X.

The aim of the first part of this thesis, consisting of Chapters I–III, is to construct the normal cycle of a compact subanalytic set. More generally, we will show that every compactly supported constructible function (i.e. a finite linear combination with integer coefficients of characteristic functions of compact subanalytic sets) admits a unique normal cycle and that, vice versa, each compactly supported subanalytic Legendrian cycle is the normal cycle of a unique compactly supported constructible function.

The first main step (Theorem I.4.2) is to show that the support function of a compactly supported constructible function is Lipschitz continuous with respect to the flat distance on $\mathbb{Z}[\mathbb{R}]$. Conversely, every continuous, subanalytic, homogeneous and Lipschitz continuous function $\mathbb{R}^n \to \mathbb{Z}[\mathbb{R}]$ is the support function of some compactly supported constructible function.

The second main step consists in proving a similar statement for a compactly supported subanalytic Legendrian cycle T, namely that its support function (which is a priori only defined almost everywhere) can be extended to a Lipschitz continuous function $\mathbb{R}^n \to \mathbb{Z}[\mathbb{R}]$ (Theorem II.4.1). It follows from the first step that T is the normal cycle of some compactly supported constructible function.

The other direction is more involved and constitutes the heart of the third chapter. Given a Lipschitz continuous, homogeneous and subanalytic function $h : \mathbb{R}^n \to \mathbb{Z}[\mathbb{R}]$, we will construct a compactly supported subanalytic Legendrian cycle the support function of which is h. Adapting arguments of J. Fu to the subanalytic situation, we give a short proof of uniqueness. Then we establish some properties of the normal cycle, some of which were not stated explicitly before.

In the second part, consisting of Chapter IV, we use the normal cycle to study a sequence of tensor-valued measures $\Lambda_{k,d}(X, -)$ associated to a compact subanalytic set X. They have several striking features, notably their symmetry and flatness properties, and can be used to generalize some notions of Riemannian geometry (scalar curvature, Einstein tensor, curvature tensor) to the setting of subanalytic sets. The measures $\Lambda_{k,0}(X, -)$ are well-known, they are the *Lipschitz-Killing measures of* X. Historically, the construction of the normal cycle by P. Wintgen, M. Zähle and J. Fu was motivated by defining these measures for several classes of singular spaces.

Throughout this work, the reader is only assumed to have a minimal knowledge of Differential Calculus and Geometry. All important notions, like o-minimal systems, definable current, flat topology, are introduced. The construction of the normal cycle only uses very basic properties of subanalytic sets. Indeed, besides finiteness properties, only C^2-cell decompositions are used. Since the existence of C^2-cell decompositions can be shown in the wider context of *o-minimal systems* and *definable sets*, our theorems can be formulated and proved not only for subanalytic sets, but for definable sets. There are several

good introductions to the theory of o-minimal systems, for instance [**18**], [**19**], [**17**].
There is just one exception, which is the proof of Theorem III.3.2, where we will use without further explication the local conical structure of definable sets and Thom's isotopy lemma. Since this theorem is not needed elsewhere in this work, we do not give any details.

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CHAPTER I

Support functions of constructible functions

1. O-minimal systems

Definition 1.1. *An o-minimal system is a collection $\mathcal{M} = (\mathcal{M}_n)$, $n \in \mathbb{Z}$, where each $\mathcal{M}_n$ is a Boolean subalgebra of the powerset of $\mathbb{R}^n$ such that the following axioms are satisfied:*

a) *algebraic subsets of $\mathbb{R}^n$ belong to $\mathcal{M}_n$;*
b) *if $X \in \mathcal{M}_n, Y \in \mathcal{M}_m$ then $X \times Y \in \mathcal{M}_{n+m}$;*
c) *if $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ denotes the projection on the first n coordinates and $X \in \mathcal{M}_{n+1}$, then $\pi(X) \in \mathcal{M}_n$;*
d) *$\mathcal{M}_1$ consists precisely of finite unions of points and intervals.*

Example.

a) Let $\mathcal{M}_n$ be the set of semialgebraic subsets of $\mathbb{R}^n$, i.e. Boolean combinations of sets of the form
$$\{x \in \mathbb{R}^n : f(x) > 0\}$$
with f a real polynomial in n variables. Axiom a) follows since $\{x \in \mathbb{R}^n : f(x) = 0\}$ is the complement of the set $\{x \in \mathbb{R}^n : f^2(x) > 0\}$. If f is a polynomial of n variables and g a polynomial of m variables, then the product of the sets $\{x \in \mathbb{R}^n : f(x) > 0\}$ and $\{y \in \mathbb{R}^m : g(y) > 0\}$ is given by the intersection $\{(x, y) \in \mathbb{R}^{n+m} : f(x) > 0\} \cap \{(x, y) \in \mathbb{R}^{n+m} : g(y) > 0\}$. Axiom b) follows. Axiom c) is known as Tarski-Seidenberg theorem. Axiom d) is easy to prove. It is also easy to see that every o-minimal system contains all semialgebraic sets.
b) A subset X of a real analytic manifold M is called *semianalytic* if for each $x \in M$ there exists a neighborhood U of x and a representation of the form
$$X \cap U = \bigcup_{i=1}^{m} \{x \in U : f_i(x) = 0, g_{i,1}(x) > 0, \dots, g_{i,k_i}(x) > 0\} \cap U$$
with real analytic functions $f_i, g_{i,j}$ on U. The set X is called *subanalytic* if it is locally the projection of a relatively compact semianalytic set $Y \subset M \times N$. A set $X \subset \mathbb{R}^n$ is called *globally subanalytic* if the image of X under the embedding $\mathbb{R}^n \to \mathbb{P}^n, (x_1, \dots, x_n) \mapsto (1 : x_1 : \dots : x_n)$ is subanalytic. For instance, the graph of the sine function is subanalytic, but not globally subanalytic. Letting $\mathcal{M}_n$ be the set of globally

subanalytic sets of $\mathbb{R}^n$ defines an o-minimal structure $\mathcal{R}_{an}$. We refer to [**11**] for properties of subanalytic and semianalytic sets.

c) The exponential function does not belong to $\mathcal{R}_{an}$, since its graph is only locally analytic but not globally subanalytic. However, there exists an o-minimal structure $\mathcal{R}_{an,\exp}$ containing subanalytic sets and the graph of the exponential map. See [**19**] for details.

In the following, we will fix an o-minimal system $\mathcal{M}$. By a *definable set* we mean a set $X \subset \mathbb{R}^n$ which belongs to $\mathcal{M}_n$.

Definition 1.2. *Let $D \subset \mathbb{R}^n$ be definable. A function $f : D \to \mathbb{R}^m$ is called definable if its graph is a definable subset of $\mathbb{R}^n \times \mathbb{R}^m$.*

From Axiom c) we infer that the image of a definable set under a definable function is again definable.

Definition 1.3. *Let $k \in \mathbb{N}$. A definable C^k-cell decomposition of $\mathbb{R}$ is a partition of $\mathbb{R}$ in finitely many cells, which are points (of dimension 0) or open intervals (dimension 1).*
A definable C^k-cell decomposition of $\mathbb{R}^n, n > 1$ is given by a C^k-cell decomposition of $\mathbb{R}^{n-1}$ and, for each cell D of $\mathbb{R}^{n-1}$, finitely many definable C^k-functions

$$\xi_{D,1} < \cdots < \xi_{D,l(d)} : D \to \mathbb{R}.$$

The cells are the graphs (of dimension $\dim D$)

$$\{(x, \xi_{D,i}(x)) \in \mathbb{R}^{n-1} \times \mathbb{R} : x \in D\}, \quad i = 1, \ldots, l(D)$$

and the (open) bands of dimension $\dim D + 1$

$$\{(x, y) \in \mathbb{R}^n \times \mathbb{R} : x \in D, \xi_{D,i}(x) < y < \xi_{D,i+1}(x)\}, \quad i = 0, \ldots, l(D)$$

where $\xi_{D,0} = -\infty, \xi_{D,l(D)+1} = \infty$.

Theorem 1.4. **C^k-cell decomposition of definable sets**
Given finitely many definable subsets $X_1, \ldots, X_m$ of $\mathbb{R}^n$ and $k \in \mathbb{Z}$, there exists a definable C^k-cell decomposition of $\mathbb{R}^n$ compatible with $X_i, i = 1, \ldots, m$ (i.e. each such set is a union of cells).

We refer to [**17**] for the proof.
Remark. The notion of a C^k-cell depends on an ordering of the coordinates. It will be convenient to use cells in a slightly more general sense, namely we will call C^k-cell decomposition any image of a C^k-cell decomposition in the above sense under an orthogonal linear map. If $W \subset \mathbb{R}^n$ is the image of the linear subspace generated by the first m coordinate lines and $\pi_W : \mathbb{R}^n \to W$ the orthogonal projection, then such a cell decomposition is *compatible* with π_W in the sense that images of cells in $\mathbb{R}^n$ under π_W are cells in W, preimages of cells in W are unions of cells of $\mathbb{R}^n$ and π_W, restricted to cells, is submersive. Theorem 1.4 implies that, given an arbitrary linear subspace W and definable

subsets $X_1, \dots, X_m$ of $\mathbb{R}^n$, there exists a C^k-cell decomposition (in the wider sense) compatible with π_W and $X_i, i = 1, \dots, m$.

Definition 1.5. *A function $\phi : \mathbb{R}^n \to \mathbb{Z}$ is called constructible if the range of ϕ is finite and $\phi^{-1}(a)$ is definable for $a \in \mathbb{Z}$. A function $\phi : X \to \mathbb{Z}$ on a definable set $X \subset \mathbb{R}^n$ is called constructible, if its extension by* 0 *is constructible.*

A definable subset $X \subset \mathbb{R}^n$ can be identified with its characteristic function, which is constructible. The restriction of ϕ to a definable subset X will be denoted by $\phi \cap X$.

2. Euler integration

Definition and Proposition 2.1. *Let $X \subset \mathbb{R}^n$ be definable. Choose a C^0-cell decomposition of $\mathbb{R}^n$ such that X is a union of cells. Then the number*

$$\chi(X) := \sum_{D \subset X} (-1)^{\dim D}$$

is independent of the choice of the cell decomposition and called Euler characteristic of X. The Euler characteristic of a constructible function $\phi : V \to \mathbb{Z}$ is defined by

$$\chi(\phi) := \sum_{a \in \mathbb{Z}} a \chi(\phi^{-1}(a)).$$

We will also write $\int_{\mathbb{R}^n} \phi(x) d\chi(x)$ instead of $\chi(\phi)$ and $\int_X \phi(x) d\chi(x)$ instead of $\chi(\phi \cap X)$.

In fact, $\chi(X)$ is the Euler characteristic with respect to Borel-Moore homology. It is not difficult to show that $\chi(X \cup Y) + \chi(X \cap Y) = \chi(X) + \chi(Y)$, which implies also that $\chi(\phi + \psi) = \chi(\phi) + \chi(\psi)$ for constructible functions ϕ and ψ. Therefore, one can think of χ as an integral.

Theorem 2.2. **Fubini for Euler characteristic**
Let $X \subset \mathbb{R}^n$ be definable and $\phi : X \to \mathbb{Z}$ be a constructible function. Given a definable function $f : X \to \mathbb{R}^m$, the push-forward $f_* \phi$, defined by

$$f_* \phi(y) := \chi(f^{-1}(y) \cap \phi), \quad y \in \mathbb{R}^m$$

is a constructible function on $\mathbb{R}^m$. Moreover,

$$\int_X \phi(x) d\chi(x) = \int_{\mathbb{R}^m} f_* \phi(y) d\chi(y).$$

The proof is easy using a cell-decomposition of the graph of f.

Corollary 2.3. *Let $\phi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{Z}$ be constructible. Then*

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \phi(x, y) d\chi(x) d\chi(y) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \phi(x, y) d\chi(y) d\chi(x).$$

DEFINITION 2.4. *The convolution of two constructible functions ϕ and ψ on $\mathbb{R}^n$ is the constructible function $\phi * \psi$ defined by*

$$\phi * \psi(x) := \int_{\mathbb{R}^n} \phi(y)\psi(x-y)d\chi(y).$$

The set of constructible functions on $\mathbb{R}^n$, endowed with addition $+$ and multiplication $*$, is a commutative ring with unit $1_{\{0\}}$.

DEFINITION 2.5. *Let ϕ be a constructible function on $\mathbb{R}^n$ and ψ a constructible function on $\mathbb{R}^m$. Then the exterior product $\phi \otimes \psi$ is the constructible function on $\mathbb{R}^n \times \mathbb{R}^m$ defined by*

$$\phi \otimes \psi(x_1, x_2) = \phi(x_1)\psi(x_2) \quad x_1 \in \mathbb{R}^n, x_2 \in \mathbb{R}^m.$$

3. Support functions

The group ring $\mathbb{Z}[\mathbb{R}]$ is the set of finite linear combinations $\sum_{i=1}^k a_i \delta_{r_i}$, where $a_i \in \mathbb{Z}$ and $r_i \in \mathbb{R}$. The sum of two such elements is defined in the obvious way, and the multiplication is given by the convolution product:

$$\left(\sum_{i=1}^k a_i \delta_{r_i}\right) \cdot \left(\sum_{j=1}^l b_j \delta_{s_j}\right) = \sum_{i=1}^k \sum_{j=1}^l a_i b_j \delta_{r_i+s_j}.$$

Elements of $\mathbb{Z}[\mathbb{R}]$ can be considered as integer multiplicity rectifiable 0-currents on $\mathbb{R}$ (compare [**20**]). If $T = \sum_{i=1}^k a_i \delta_{r_i}$ and $f \in C_c^\infty(\mathbb{R})$, then $T(f) := \sum_{i=1}^k a_i f(r_i)$. The augmentation of T is the integer $T(1) = \sum_{i=1}^k a_i$. We can identify $\mathbb{R}$ with a subset of $\mathbb{Z}[\mathbb{R}]$ by sending x to δ_x. Sometimes it will be useful to consider an element $T = \sum_i a_i \delta_{r_i}$ of $\mathbb{Z}[\mathbb{R}]$ as a constructible function on $\mathbb{R}$, which equals a_i at r_i and 0 otherwise (here we assume that all r_i are pairwise different).

PROPOSITION 3.1. *Let ϕ be a constructible function on $\mathbb{R}$.*

a)
$$\sum_{x \in \mathbb{R}} \lim_{s \to 0^+} (\phi(x) - \phi(x+s))\delta_x$$
is an element of $\mathbb{Z}[\mathbb{R}]$, denoted by ϕ' and called jump of ϕ.

b) *If ϕ has compact support, then*
$$\phi'(1) = \chi(\phi).$$

c) *If ϕ is continuous from the left, then*
$$\int_{\mathbb{R}} \phi(s)d\chi(s) = -\lim_{s\to\infty} \phi(s).$$

PROOF. Since ϕ is constructible, there exists a finite partition of $\mathbb{R}$ into points and open intervals, such that ϕ is constant on each cell. If x belongs to an open interval, then the coefficient before δ_x vanishes from which a) follows. Statement b) is easily verified. If ϕ is continuous from the left, then ϕ is constant on finitely many half-open intervals of the from $(a, b]$ (where $a = -\infty$ is possible) and on one open interval (a, ∞). Since $\chi((a, b]) = 0, \chi((a, \infty)) = -1$, c) follows. $\square$

EXAMPLE. If ϕ is the characteristic function of a compact interval $[a, b]$, then $\phi' = \delta_b$.
In the following, V denotes an n-dimensional Euclidean vector space. After choice of an orthogonal basis, V can be identified with $\mathbb{R}^n$. The notions *definable subset* and *constructible function* are independent of the choice of this basis.

DEFINITION 3.2. *Let ϕ be a constructible function on V. For each $y \in V$ let $\pi_y : V \to \mathbb{R}, x \mapsto \langle x, y \rangle$ and define*

$$h_\phi(y) := ((\pi_y)_* \phi)' \in \mathbb{Z}[\mathbb{R}].$$

The function

$$h_\phi : V \to \mathbb{Z}[\mathbb{R}]$$

is called support function of ϕ.

PROPOSITION 3.3. a) $h_\phi(0) = \chi(\phi)\delta_0$.

b) *h_ϕ is homogeneous in the following sense: if $\lambda \geq 0$, then*

$$h_\phi(\lambda y) = (m_\lambda)_*(h_\phi(y)),$$

where $(m_\lambda)_*(\sum_i a_i \delta_{r_i}) := \sum_i a_i \delta_{\lambda r_i}$.

c) *If ϕ has compact support, then the augmentation of $h_\phi(y)$ equals $\chi(\phi)$ for all $y \in V$.*

PROPOSITION 3.4. a) *Let $A \in GL(V)$. Given a constructible function ϕ on V, define $A_*\phi$ by $A_*\phi(x) := \phi(A^{-1}x)$. Then*

$$h_{A_*\phi}(y) = h_\phi(A^* y).$$

b) *For constructible functions ϕ and ψ on V,*

$$\begin{aligned} h_{\phi+\psi} &= h_\phi + h_\psi \\ h_{\phi * \psi} &= h_\phi \cdot h_\psi. \end{aligned}$$

c) *Let W be a Euclidean vector space. For a constructible function ϕ on V and a constructible function ψ on W,*

$$h_{\phi \otimes \psi}(y_1, y_2) = h_\phi(y_1) \cdot h_\psi(y_2) \quad \forall y_1 \in V, y_2 \in W.$$

PROOF. Using Fubini's Theorem, the proofs are easy. $\square$

Example.

- Let $K \subset V$ be a compact (definable) convex set and $\phi := 1_K$ its characteristic function. If $y \in V$, then the push-forward $(\pi_y)_*\phi$ is the characteristic function of the compact interval $[\min_{x\in K}\langle x,y\rangle, \max_{x\in K}\langle x,y\rangle]$. The jump of this function is given by $\delta_{\max_{x\in K}\langle x,y\rangle}$. The function mapping y to $\max_{x\in K}\langle x,y\rangle$ is the classical support function of K (compare [**39**]). Therefore, the support function of 1_K is the same (using the embedding $\mathbb{R} \to \mathbb{Z}[\mathbb{R}]$) as the classical support function of K.
- Let ϕ be the characteristic function of $S(V) := \{x \in V : \|x\| = 1\}$. Then $(\pi_y)_*\phi(t)$ equals 1 for $|t| = \|y\|$, 0 for $|t| > \|y\|$ and $\chi(S^{n-2}) = 1 + (-1)^n$ for $\|t\| < \|y\|$. It follows that
$$h_\phi(y) = \delta_{\|y\|} + (-1)^{n+1}\delta_{-\|y\|}.$$

Proposition 3.5. *Let ϕ be a constructible function on V. Let W be a linear subspace and $\pi : V \to W$ the orthogonal projection. Then $\pi_*\phi$ is a constructible function on W and*
$$h_{\pi_*\phi} = h_\phi|_W.$$

Proof. Immediate from Fubini's Theorem 2.2. □

Definition 3.6. *A function $h : \mathbb{R}^n \to \mathbb{Z}[\mathbb{R}]$ is called definable if the function*
$$\mathbb{R}^n \times \mathbb{R} \to \mathbb{Z}, (y,r) \mapsto h(y)(r)$$
is constructible.

Proposition 3.7. *If $\phi : V \to \mathbb{Z}$ is constructible, then $h_\phi : V \to \mathbb{Z}[\mathbb{R}]$ is definable.*

Proof. Again, this is an easy consequence of Theorems 1.4 and 2.2. □

Theorem 3.8. *A function $h : V \to \mathbb{Z}[\mathbb{R}]$ is the support function of a constructible function ϕ on V if and only if h is definable and homogeneous.*

The "only if"-part is contained in Proposition 3.3, b) and Proposition 3.7. In [**12**] one finds the proof of the "if"-part. In the next section we prove a similar statement.

4. Constructible functions with compact support

Definition 4.1. *The flat norm of an element $T = \sum_{i=1}^k a_i\delta_{r_i} \in \mathbb{Z}[\mathbb{R}]$ is defined by*
$$\mathbf{F}(T) := \sup\{T(f) : f \in C_c^\infty(\mathbb{R}) : \|f\|_\infty \leq 1, \|f'\|_\infty \leq 1\}.$$
The mass of T is defined by
$$\mathbf{M}(T) = \sum_{i=1}^k |a_i|.$$

THEOREM 4.2. *A function* $h : V \to \mathbb{Z}[\mathbb{R}]$ *is the support function of a compactly supported constructible function* ϕ *on* V *if and only if* h *is definable, homogeneous and Lipschitz with respect to* $\mathbf{F}$. *In this case,* ϕ *is unique.*

PROOF. **Compact support implies Lipschitz**
Suppose $h = h_\phi$ is the support function of a constructible function ϕ on V. By Proposition 3.3, b) and Proposition 3.7, h is homogeneous and definable.
Suppose that the support of ϕ is contained in a compact set, say $\operatorname{spt}\phi \subset B(0,R), R > 0$. Since h is definable, there is an $M > 0$ with

$$\mathbf{M}(h(y)) \leq M \quad \forall y \in V.$$

We claim that h is $6MR$-Lipschitz with respect to $\mathbf{F}$. It is enough to show that the restriction of h to each two-dimensional linear subspace W is $6MR$-Lipschitz. Since $h|_W$ is the support function of $\pi_*\phi$, with $\pi : V \to W$ orthogonal projection, and since $\pi_*\phi$ has support in $B(0,R) \subset W$, it is enough to suppose from the start that $\dim V = 2$. Thus we assume that ϕ is a constructible function on V, $\dim V = 2$, $\operatorname{spt}\phi \subset B(0,R)$, $\mathbf{M}(h(y)) \leq M, y \in V$ and we have to prove that h is $6MR$-Lipschitz. We can furthermore assume that ϕ is not constantly 0.
A standard argument shows that it suffices to prove that every $y \in V$ has a neighborhood U such that $\mathbf{F}(h(y') - h(y)) \leq 6MR\|y' - y\|$ for all $y' \in U$.
We fix an orthogonal basis of V and identify V with $\mathbb{R}^2$.
Suppose first that $y = 0$. Then $h(0) = \chi(\phi)\delta_0$ by 3.3 a); $h(y') = \sum_{i=1}^k a_i\delta_{r_i}$ with $\sum_{i=1}^k a_i = \chi(\phi)$ (3.3 c)), $\sum_{i=1}^k |a_i| \leq M$ and $|r_i| \leq R\|y'\|$.
Then

$$\mathbf{F}\left(\sum_{i=1}^k a_i\delta_{r_i} - \chi(\phi)\delta_0\right) \leq \sum_{i=1}^k |a_i|\mathbf{F}(\delta_{r_i} - \delta_0) \leq \sum_{i=1}^k |a_i|R\|y'\| \leq MR\|y'\|.$$

Next we suppose that $y \neq 0$. Using homogeneity, we can assume without loss of generality that $y = (1,0)$.
By Theorem 1.4, there exists a C^2-cell decomposition of $\mathbb{R}^2$ such that ϕ is constant on each cell. We can refine the decomposition and assume that each of the functions $\xi_{D,i}, i = 1, \ldots, l(D)$, where D runs over the cells of $\mathbb{R}$, is convex or concave.

LEMMA 4.3. *If* $\xi : I \to \mathbb{R}$ *is a convex or concave* C^2*-function on a bounded open interval* $I \subset \mathbb{R}$ *such that* $\operatorname{graph}(\xi) \subset B(0,R)$, *then for* $s \in I$

$$|\xi'(s)| \leq \frac{2R}{d(s,\partial I)}.$$

PROOF. Assume $I = (a,b), s \in I$ and ξ is convex (the concave case follows by applying the convex case to $-\xi$).

Let $\Psi : \mathbb{R} \to \mathbb{R}$ denote a supporting function at s, i.e. Ψ is affine, $\Psi|_I \leq \xi$, $\Psi(s) = \xi(s)$ and $\Psi'(s) = \xi'(s)$.
For $t \in (s, b)$, $\Psi'(s) = \frac{\Psi(t)-\Psi(s)}{t-s} \leq \frac{\xi(t)-\xi(s)}{t-s} \leq \frac{2R}{t-s}$. Letting t tend to b we obtain $\Psi'(s) \leq \frac{2R}{d(s,\partial I)}$. The equality $\Psi'(s) \geq -\frac{2R}{d(s,\partial I)}$ is proved in a similar way. $\square$

Since ϕ has compact support, it is non-zero only on finitely many, bounded cells. Fix a number $0 < \rho_{max} < \frac{1}{2}$ such that $12R\rho_{max}$ is smaller than the lengths of the cells in $\mathbb{R}$ above which ϕ is non-zero.

LEMMA 4.4. *Let $y' = (y_1', y_2') \in \mathbb{R}^2$ with $\rho := \|y - y'\| < \rho_{max}$. Let $t \in D$, where D is an open cell of $\mathbb{R}$ and suppose $d(t, \partial D) > \epsilon := 6\rho R$. Then each intersection of the line $L_t = L_t(y')$ defined by*

$$L_t = \{(x_1, x_2) \in \mathbb{R}^2 : y_1'x_1 + y_2'x_2 = t\}$$

with a cell contained in $B(0, R)$ is empty or transversal.

PROOF. Let $D = (a, b)$, where $a = -\infty$ or $b = \infty$ is possible.
Let $(x_1, x_2) \in B(0, R)$ be the intersection of L_t with a cell.
Then

$$\rho R \geq |y_2'x_2| = |t - x_1 + x_1 - y_1'x_1| \geq |t - x_1| - |x_1(1 - y_1')| \geq |t - x_1| - \rho R,$$

which implies $|t - x_1| \leq 2\rho R = \frac{\epsilon}{3}$.
It follows that $x_1 \in D$ and $d(x_1, \partial D) > \frac{2}{3}\epsilon$.
The cell in which (x_1, x_2) lies is either a band or a graph of a function $\xi = \xi_{D,i}$. In the first case, the intersection is trivially transversal.
In the second case,

$$|\xi'(x_1)| \leq \frac{2R}{d(x_1, \partial D)} < \frac{3R}{\epsilon} = \frac{1}{2\rho}.$$

If the intersection is non-transversal, then $y_2' \neq 0$ and

$$\frac{1}{2\rho} \leq \frac{|y_1'|}{|y_2'|} = |\xi'(x_1)| < \frac{1}{2\rho},$$

a contradiction. $\square$

Let $r_1 < \ldots < r_k$ be the endpoints of the cells of $\mathbb{R}$. Denote by π the projection of $\mathbb{R}^2$ to the first coordinate. Since the Euler characteristic of $\pi^{-1}(s) \cap \phi, s \in \mathbb{R}$ is constant on each cell, $h_\phi(y)$ is concentrated on $\{r_1, \ldots, r_k\}$, say $h_\phi(y) = \sum_{i=1}^k a_i \delta_{r_i}$.
Denote $\pi' : \mathbb{R}^2 \to \mathbb{R}, x \mapsto \langle x, y' \rangle$. Let $t \in \mathbb{R}$ be at distance at least ϵ from $\{r_1, \ldots, r_k\}$. By Lemma 4.4, $(\pi')^{-1}(t) = L_t(y')$ intersects the cell decomposition transversally, which implies that

$$\chi((\pi')^{-1}(t)) = \chi(\pi^{-1}(t)).$$

It follows that, if $h_\phi(y') = \sum_{j=1}^l b_j \delta_{s_j}$, then the s_j are contained in the open ϵ-neighborhood of $\{r_1, \ldots, r_k\}$. Note that the ϵ-neighborhoods of the different r_i are disjoint by choice of ϵ.

From Theorem 2.2 and Proposition 3.1, applied to the function $[r_i - \epsilon, r_i + \epsilon) \to \mathbb{Z}, s \mapsto \chi(\pi^{-1}(s) \cap \phi)$ we infer that

$$\chi(\pi^{-1}[r_i - \epsilon, r + \epsilon) \cap \phi) = a_i$$

In the same way,

$$\chi((\pi')^{-1}[r_i - \epsilon, r + \epsilon) \cap \phi) = \sum_{j:|s_j - r_i| \leq \epsilon} b_j.$$

The intersection of the strip $\pi^{-1}[r_i - \epsilon, r + \epsilon)$ with the cell decomposition is transversal by Lemma 4.4. The same is true for all y'' on the line between y and y'. By simple counting (or by applying Thom's isotopy lemma, [**28**], [**37**]), we obtain that the Euler characteristics are equal, which means that

$$\sum_{j:|s_j - r_i| \leq \epsilon} b_j = a_i.$$

Using $\mathbf{F}(\delta_t - \delta_s) \leq |s - t|$ for reals s, t, we get that

$$\begin{aligned}
\mathbf{F}(h_\phi(y') - h_\phi(y)) &= \mathbf{F}\left(\sum_{i=1}^{k} \sum_{j:|s_j - r_i| \leq \epsilon} (b_j \delta_{s_j} - b_j \delta_{r_i})\right) \\
&\leq \sum_{i=1}^{k} \sum_{j:|s_j - r_i| \leq \epsilon} \underbrace{\mathbf{F}(b_j \delta_{s_j} - b_j \delta_{r_i})}_{\leq \epsilon |b_j|} \\
&\leq \epsilon \sum_{j=1}^{l} |b_j| \\
&= \epsilon \mathbf{M}(h_\phi(y')) \\
&\leq 6MR\|y' - y\|.
\end{aligned}$$

We deduce that h is $6MR$-Lipschitz.

Remark. For later use we note the following. Let $\phi : V \to \mathbb{Z}$ be a constructible function with support in $B(0, R)$. Let $D \subset V$ be a C^2-cell such that $h = h_\phi$ is given on D by

$$h(y) = \sum_{i=1}^{k} a_i \delta_{f_i(y)}$$

with definable C^2-functions $f_i : D \to \mathbb{R}, f_1 < \ldots < f_k$ and non-zero natural numbers a_i. Then the above argument shows that the norm of the gradient of each f_i is bounded by $6R$.

Uniqueness

Let $h = h_\phi$ be the support function of a compactly supported constructible function ϕ. Then $\chi(\phi) = h(y)(1)$ for all $y \in V$, in particular $\chi(\phi)$ can be computed from h alone.

We compute that for all $t \in \mathbb{R}$

$$h(y)(-\infty, t) = \int_V \phi(x) 1_{\{x' \in V : \langle x', y \rangle < t\}}(x) d\chi(x).$$

Since h is definable, the function

$$\psi_x(y) := h(y)(-\infty, \langle x, y \rangle)$$

is constructible.
Then

$$\begin{aligned}
\int_V \psi_x(y) d\chi(y) &= \int_V h(y)(-\infty, \langle x, y \rangle) d\chi(y) \\
&= \int_V \int_V \phi(z) 1_{\{x' \in V : \langle x', y \rangle < \langle x, y \rangle\}}(z) d\chi(z) d\chi(y) \\
&= \int_V \phi(z) \int_V 1_{\{x' \in V : \langle x' - x, y \rangle < 0\}}(z) d\chi(y) d\chi(z) \\
&= \int_V \phi(z) \underbrace{\chi(\{y \in V : \langle z - x, y \rangle < 0\})}_{=0 \text{ if } z = x, (-1)^n \text{ else}} d\chi(z) \\
&= (-1)^n (\chi(\phi) - \phi(x)).
\end{aligned}$$

It follows that

$$\phi(x) = \chi(\phi) + (-1)^{n-1} \int_V \psi_x(y) d\chi(y)$$

is uniquely determined by h. This holds true for all $x \in V$, therefore ϕ is unique.

Lipschitz implies compact support
Suppose that $h : V \to \mathbb{Z}[\mathbb{R}]$ is definable, homogeneous and Lipschitz with Lipschitz constant $L > 0$. We will show that h is the support function of a constructible function with support in $B(0, L)$.

Step 1: We claim that $\operatorname{spt} h(y) \subset [-L\|y\|, L\|y\|]$ for all $y \in V$. To prove the claim, fix $y \in V$ and suppose that $h(y) = \sum_{i=1}^k a_i \delta_{r_i}$ with $a_i \in \mathbb{Z}, a_i \neq 0$ and $r_1 < r_2 < \ldots < r_k$. Suppose $r_k > 0$.
Fix a real number $\lambda > 1$ such that $\lambda r_{k-1} < r_k$ and $(\lambda - 1) r_k < 1$. Let f be a piecewise affine function which equals 0 on $(-\infty, r_k]$, grows linearly on $[r_k, \lambda r_k]$, equals 1 at λr_k and which is symmetric with respect to the center λr_k. By homogeneity, $h(\lambda y) = \sum_{i=1}^k a_i \delta_{\lambda r_i}$ and therefore $h(\lambda y)(f) = a_k, h(y)(f) = 0$. Approximating f by compactly supported smooth functions and using that h is L-Lipschitz with respect to $\mathcal{F}$, we obtain

$$|a_k| = |h(\lambda y)(f) - h(y)(f)| \leq L \|\lambda y - y\| \max\{\underbrace{\|f\|_\infty}_{=1}, \underbrace{\|f'\|_\infty}_{=\frac{1}{(\lambda-1) r_k}}\} = \frac{L\|y\|}{r_k}.$$

We deduce $r_k \leq L\|y\|$ and similarly $r_1 \geq -L\|y\|$.

Step 2: With h being Lipschitz, the value $a := h(y)(1) \in \mathbb{Z}$ is independent of $y \in V$.
Since h is definable, the function

$$\psi_x(y) := h(y)(-\infty, \langle x, y \rangle)$$

is constructible.
We have seen in the uniqueness proof that if there exists ϕ with $h_\phi = h$, then ϕ has to be given by

$$\phi(x) := a + (-1)^{n-1} \int_V \psi_x(y) d\chi(y).$$

We claim that indeed $h_\phi = h$.
Given $v_0 \in S(V)$ and $t_0 \in \mathbb{R}$, we set

$$W_0 := \{x \in V : \langle x, v_0 \rangle = t_0\}.$$

Fubini's theorem shows that

$$\int_{W_0} \int_V \psi_x(y) d\chi(y) d\chi(x) = \int_V \int_{W_0} \psi_x(y) d\chi(x) d\chi(y).$$

We evaluate the inner integral and consider several cases for y.

a) $y = 0$.
By homogeneity, $h(0) = a\delta_0$ and thus $\psi_x(0) = 0$ for all x. It follows

$$\int_{W_0} \psi_x(0) d\chi(x) = 0.$$

b) $y \| v_0, y \neq 0$.
For $\lambda > 0$ we obtain

$$\begin{aligned} \psi_x(\lambda v_0) &= h(\lambda v_0)(-\infty, \langle x, \lambda v_0 \rangle) = h(v_0)(-\infty, t_0) \\ \psi_x(-\lambda v_0) &= h(-v_0)(-\infty, -t_0). \end{aligned}$$

and thus

$$\begin{aligned} \int_{W_0} \psi_x(\lambda v_0) d\chi(x) &= (-1)^{n-1} h(v_0)(-\infty, t_0) \\ \int_{W_0} \psi_x(-\lambda v_0) d\chi(x) &= (-1)^{n-1} h(-v_0)(-\infty, -t_0). \end{aligned}$$

c) $y \not\| v_0$.
With $y^\perp := y - \langle y, v_0 \rangle v_0$ we get $\langle y^\perp, v_0 \rangle = 0$ and $\langle y^\perp, y \rangle > 0$.

For $x_0 \in W_0$, the line $l := \{x_0 + sy^\perp : s \in \mathbb{R}\}$ is contained in W_0 and we compute, using Proposition 3.1, c),

$$\begin{aligned}\int_l \psi_x(y)d\chi(x) &= \int_{\mathbb{R}} \psi_{x_0+sy^\perp}(y)d\chi(s)\\ &= \int_{\mathbb{R}} h(y)(-\infty, \langle x_0 + sy^\perp, y\rangle)d\chi(s)\\ &= \int_{\mathbb{R}} h(y)(-\infty, s')d\chi(s')\\ &= -a.\end{aligned}$$

This is true for all lines in W_0 parallel to $y^\perp$ and implies, by Fubini's theorem,

$$\int_{W_0} \psi_x(y)d\chi(x) = (-1)^{n-1}a.$$

From these considerations, we deduce that

$$\int_V \int_{W_0} \psi_x(y)d\chi(x)d\chi(y) = (-1)^n\big(h(v_0)(-\infty, t_0)+ \\ + h(-v_0)(-\infty, -t_0)\big) + (1 + (-1)^n)(-1)^{n-1}a.$$

It follows that

$$\int_{W_0} \phi(x)d\chi(x) = a - h(v_0)(-\infty, t_0) - h(-v_0)(-\infty, -t_0),$$

from which we easily deduce that $h_\phi = h$ on $S(V)$, and then, by homogeneity of both sides, on V.
Step 3:

LEMMA 4.5. *Let $D \subset V$ be a C^1-cell and let $h : D \to \mathbb{Z}[\mathbb{R}]$ be given by $h(y) = \sum_{i=1}^k a_i\delta_{f_i(y)}$ with real-valued C^1-functions $f_1 < \ldots < f_k$ on D and non-zero integers a_i. If h is L-Lipschitz with respect to $\mathbf{F}$, then $\|\operatorname{grad} f_i(y)\| \le L$ for all $y \in D$ and $i = 1, \ldots, k$.*

PROOF. Fix $y \in D$ and $i \in \{1, \ldots, k\}$ and set $c := f_i(y)$. Let $1 > \eta > 0$ be smaller than the minimum of $f_{i+1}(y) - f_i(y)$ and $f_i(y) - f_{i-1}(y)$ (if $i = 1$ or $i = k$ or even $k = 1$ then the corresponding difference will be set to be ∞).
By continuity of the f_i, there exists a neighborhood $U \subset D$ of y such that $f_{i+1}(y') > c + \eta$, $f_{i-1}(y') < c - \eta$ and $c - \eta \le f_i(y') \le c + \eta$ for $y' \in U$.
Define a piecewise affine function g on $\mathbb{R}$ by

$$g(z) = \begin{cases} 1 - \frac{|z-c|}{\eta} & z \in [c - \eta, c + \eta] \\ 0 & \text{otherwise.}\end{cases}$$

Since h is L-Lipschitz and since $\|g\|_\infty = 1 < \frac{1}{\eta}$, $\|g'\|_\infty = \frac{1}{\eta}$, we get for all $y' \in U$

$$\frac{|a_i|}{\eta}|f_i(y') - f_i(y)| = |a_i g(f_i(y')) - a_i g(f_i(y))| = |h(y)(g) - h(y')(g)|$$
$$\leq \frac{L}{\eta}\|y' - y\|.$$

It follows that $\| \operatorname{grad} f_i\| \leq \frac{L}{a_i} \leq L$. □

Step 4: Let $x \in V$ with $t := \|x\| > L$. Write $x = tv_0$ with $v_0 \in S(V)$ and fix $y_0 \in V$.
We claim that the function $g : \mathbb{R} \to \mathbb{Z}$ defined by

$$g(s) := h(y_0 + sv_0)(-\infty, \underbrace{\langle x, y_0 + sv_0\rangle}_{\langle x,y_0\rangle + st})$$

is continuous from the left.
The function $s \mapsto h(y_0+sv_0)$ is definable and L-Lipschitz by hypothesis. Fix $s_0 \in \mathbb{R}$. For all s in some interval $(s_0 - \epsilon, s_0)$, we get

$$h(y_0 + sv_0) = \sum_{i=1}^k a_i \delta_{f_i(s)},$$

with definable, real-valued C^1-functions $f_1 < \ldots < f_k$. Lemma 4.5 implies that they are L-Lipschitz. Each f_i is bounded (compare Step 1) and can be extended by continuity to s_0. Then we also have $h(y_0 + s_0 v_0) = \sum_{i=1}^k a_i \delta_{f_i(s_0)}$.
It follows that

$$g(s) = \sum_{i : f_i(s) < \langle x, y_0\rangle + st} a_i$$

for all $s \in (s_0 - \epsilon, s_0]$.
If $f_i(s_0) \neq c := \langle x, y_0\rangle + s_0 t$, then either $f_i(s) < \langle x, y_0\rangle + st$ or $f_i(s) > \langle x, y_0\rangle + st$ for all s in some (maybe smaller) interval $(s_0 - \epsilon, s_0]$.
If $f_i(s_0) = c$, then, since f_i is L-Lipschitz, for all $s < s_0$ near s_0

$$f_i(s) \geq f_i(s_0) + L(s - s_0) = \langle x, y_0\rangle + st + (t - L)(s_0 - s) \geq \langle x, y_0\rangle + st.$$

We deduce that $g(s) = g(s_0)$ for all $s < s_0$ near s_0, which proves the claim.
Step 5:
The function g from Step 4 is continuous from the left and satisfies $\lim_{s\to\infty} g(s) = a$, since $\operatorname{spt} h(y_0 + sv_0) \subset (-\infty, \langle x, y_0 + sv_0\rangle)$ for large s (compare Step 1).
With $l := \{y_0 + sv_0 : s \in \mathbb{R}\}$, and using Proposition 3.1 c), we get

$$\int_l \psi_x(y) d\chi(y) = \int_{\mathbb{R}} g(s) d\chi(s) = -a.$$

The same holds true for every line parallel to v_0. Fubini's theorem implies that

$$\phi(x) = a + (-1)^{n-1} \int_V \psi_x(y) d\chi(y) = 0.$$

Therefore the support of ϕ is contained in $B(0, L)$. $\square$

CHAPTER II

Legendrian cycles

1. Definable currents

Let $\mathcal{E}^k(\mathbb{R}^n)$ be the set of differential k-forms on $\mathbb{R}^n$ and $\mathcal{D}^k(\mathbb{R}^n)$ the subset consisting of k-forms with compact support. For a multi-index $I = (i_1, \ldots, i_k)$ we write $dx_I := dx_{i_1} \wedge \ldots \wedge dx_{i_k}$. A k-form ω can be written in the form $\omega = \sum_I \omega_I dx_I$.
The topology of $\mathcal{D}^k(\mathbb{R}^n)$ is the usual one, which is characterized by the fact that a sequence $\omega^1, \omega^2, \ldots \in \mathcal{D}^k(\mathbb{R}^n)$ converges to $\omega \in \mathcal{D}^k(\mathbb{R}^n)$ if and only if there is a compact set $K \subset \mathbb{R}^n$ such that the supports of $\omega^1, \ldots$ are contained in K and such that all partial derivatives (of arbitrary degree) of the coefficients ω_I^j converge uniformly to the corresponding derivatives of ω_I.

Definition 1.1. *A continuous functional $T : \mathcal{D}^k(\mathbb{R}^n) \to \mathbb{R}$ is called a (Federer-Fleming-) k-current on $\mathbb{R}^n$. The space of k-currents is denoted by $\mathcal{D}_k(\mathbb{R}^n) = \left(\mathcal{D}^k(\mathbb{R}^n)\right)^*$.*

The *boundary* ∂T of $T \in \mathcal{D}_k(\mathbb{R}^n)$ is the current $\partial T \in \mathcal{D}_{k-1}(\mathbb{R}^n)$ defined by

$$\partial T(\omega) = T(d\omega) \qquad \forall \omega \in \mathcal{D}^{k-1}(\mathbb{R}^n).$$

T is called a *cycle* if $\partial T = 0$.
The *restriction* of $T \in \mathcal{D}_k(\mathbb{R}^n)$ to a form $\xi \in \mathcal{D}^l(\mathbb{R}^n), l \leq k$ is the current $T \llcorner \xi \in \mathcal{D}^{k-l}(\mathbb{R}^n)$ with

$$T \llcorner \xi(\omega) = T(\xi \wedge \omega) \qquad \forall \omega \in \mathcal{D}^{k-l}(\mathbb{R}^n).$$

The *support* of $T \in \mathcal{D}_k(\mathbb{R}^n)$ is the closed set

$$\operatorname{spt} T := \left\{ \bigcap_{K \subset \mathbb{R}^n} K \text{ closed } : T(\omega) = 0 \right.$$
$$\left. \text{for all } \omega \in \mathcal{D}^k(\mathbb{R}^n) \text{ with } \operatorname{spt} \omega \subset \mathbb{R}^n \setminus K \right\}.$$

Example. An oriented, k-dimensional C^1-manifold M with boundary ∂M defines a k-current $[[M]] \in \mathcal{D}_k(\mathbb{R}^n)$ by

$$[[M]](\omega) = \int_M \omega, \qquad \omega \in \mathcal{D}^k(\mathbb{R}^n).$$

Stokes's theorem implies that

$$\partial[[M]] = [[\partial M]].$$

In the same way, the k-dimensional cells of a C^1-cell decomposition of $\mathbb{R}^n$ define k-currents whose boundaries are given by integration over $k-1$-dimensional cells.

Definition 1.2. *A current $T \in \mathcal{D}_k(\mathbb{R}^n)$ is called definable if there exists a definable C^1-cell decomposition and for each k-cell D an orientation of D and a number n_D such that*

$$T = \sum_D n_D[[D]].$$

The mass of T is defined by

$$\mathbf{M}(T) := \sum_D |n_D| \operatorname{vol}_k(D) \in [0, \infty]$$

The boundary of a definable current is again a definable current. It follows that definable currents are locally integral currents in the sense of [**20**]. In particular, they are locally normal currents, i.e. the mass and the mass of the boundary are finite on compact sets.

If $T \in \mathcal{D}_k(\mathbb{R}^n)$ and $A \subset \mathbb{R}^n$ are definable, then the current $T \llcorner A$ defined by

$$T \llcorner A = \sum_D n_D[[D \cap A]]$$

is again a definable current.

Given a definable C^1-map $f : \mathbb{R}^n \to \mathbb{R}^m$ which is proper on the support of T (i.e. $f^{-1}(K) \cap \operatorname{spt} T$ is compact whenever $K \subset \mathbb{R}^m$ is compact), the current f_*T with

$$f_*T(\omega) := T(f^*\omega)$$

is again a definable current, called *image* of T under f. Note that $f_* \circ \partial = \partial \circ f_*$.

Given definable currents $S \in \mathcal{D}_k(\mathbb{R}^n)$ and $T \in \mathcal{D}_l(\mathbb{R}^m)$, their *direct product* $S \times T \in \mathcal{D}_{k+l}(\mathbb{R}^n \times \mathbb{R}^m)$ is defined in the obvious way, i.e. $S \times T$ is defined by integration over the products of the cells of S and T, counted with the product of the multiplicities.

Proposition 1.3. **Homotopy formula**

Let $T \in \mathcal{D}_k(\mathbb{R}^n)$. Suppose $H : [0,1] \times \mathbb{R}^n \to \mathbb{R}^n$ is a definable C^1 homotopy of $\mathbb{R}^n$ between f and g such that $H^{-1}(K) \cap \operatorname{spt} T$ is compact in $[0,1] \times \mathbb{R}^n$ for any compact set $K \subset \mathbb{R}^n$. Then

$$g_*T - f_*T = \partial H_*([0,1] \times T) + H_*([0,1] \times \partial T).$$

Proof. The assumption implies that the currents $H_*(T \times [0,1])$ and $H_*(\partial T \times [0,1])$ are well defined. The formula follows from

$$\begin{aligned} \partial H_*([0,1] \times T) &= H_*\partial([0,1] \times T) \\ &= H_*((\partial[0,1]) \times T) - H_*([0,1] \times \partial T) \\ &= \underbrace{H_*(\delta_1 \times T)}_{g_*T} - \underbrace{H_*(\delta_0 \times T)}_{f_*T} - H_*([0,1] \times \partial T). \end{aligned}$$

□

For completeness, we prove a very special case of the constancy theorem ([**20**], 4.1.7):

Proposition 1.4. **Constancy theorem**
If a definable current $T \in \mathcal{D}_n(\mathbb{R}^n)$ satisfies $\partial T = 0$ and has compact support, then it vanishes.

Proof. Choose a C^1 cell decomposition of $\mathbb{R}^n$ such that $T = \sum_D n_D[[D]]$. The n-cells above an $n-1$-cell D' of $\mathbb{R}^{n-1}$ are given by open bands

$$D_i := \{(x,y) \in D' \times \mathbb{R} : \xi_{D',i}(x) < y < \xi_{D',i+1}(x)\}, \quad i = 0, \ldots, l(D'),$$

where $\xi_{D',1} < \ldots < \xi_{D',l(D')}$ are definable C^1 functions on D' and $\xi_{D',0} = -\infty, \xi_{D',l(D')+1} = \infty$.
The boundary of $[[D_i]]$ is given by integration over the graph of $\xi_{D',i+1}$ minus integration over the graph of $\xi_{D',i}$ plus some components lying above the boundary of D'. These boundaries cancel out only if $n_{D_i} = n_{D_{i+1}}$ for $i = 0, \ldots, l(D') - 1$. Since the support of T is compact, $n_{D_0} = 0$ and hence $n_{D_i} = 0$ for $i = 0, 1, \ldots, l(D')$. The same argument works above each $n-1$-cell D' of $\mathbb{R}^{n-1}$ and shows that $n_D = 0$ for all n-dimensional cells D, i.e. $T = 0$. □

2. Slicing definable currents

The proof of the next proposition is contained in [**31**], [**32**].

Proposition 2.1. *Let $T \in \mathcal{D}_k(\mathbb{R}^n)$ be a definable current and $f : \mathbb{R}^n \to \mathbb{R}^l$ a definable map. Then for all $y \in \mathbb{R}^l$ such that $\dim f^{-1}(y) \cap \operatorname{spt} T \leq k - l$ and $\dim f^{-1}(y) \cap \operatorname{spt} \partial T \leq k - l - 1$ there exists a definable current $\langle T, f, y\rangle \in \mathcal{D}_{k-l}(\mathbb{R}^n)$, called slice of T, with the following properties:*

a) $\operatorname{spt}\langle T, f, y\rangle \subset \operatorname{spt} T \cap f^{-1}(y)$;
b) $\partial\langle T, f, y\rangle = (-1)^l \langle \partial T, f, y\rangle$;
c) *if $g : \mathbb{R}^n \to \mathbb{R}^m$ is a definable function which is proper on the support of T, and $f : \mathbb{R}^m \to \mathbb{R}^l$ any definable function, then*
$$g_*\langle T, f \circ g, y\rangle = \langle g_* T, f, y\rangle$$
whenever $y \in \mathbb{R}^l$ and $\dim(f \circ g)^{-1}(y) \cap \operatorname{spt} T \leq k - l$ and $\dim(f \circ g)^{-1}(y) \cap \operatorname{spt} \partial T \leq k - l - 1$;
d) *let $f' : \mathbb{R}^n \to \mathbb{R}^{l'}$ be a definable function, then*
$$\langle\langle T, f, y\rangle, f', y'\rangle = \langle T, (f, f'), (y, y')\rangle$$
for all $y \in \mathbb{R}^l, y' \in \mathbb{R}^{l'}$ such that the following dimension restrictions holds: $\dim f^{-1}(y) \cap \operatorname{spt} T \leq k - l$, $\dim f^{-1}(y) \cap \operatorname{spt} \partial T \leq k - l - 1$, $\dim(f')^{-1}(y') \cap \operatorname{spt} T \leq k - l'$, $\dim(f')^{-1}(y') \cap \operatorname{spt} \partial T \leq k - l' - 1$, $\dim f^{-1}(y) \cap (f')^{-1}(y') \cap \operatorname{spt} T \leq k - l - l'$, $\dim f^{-1}(y) \cap (f')^{-1}(y') \cap \operatorname{spt} \partial T \leq k - l - l' - 1$;

e) *if* $g : \mathbb{R}^l \to \mathbb{R}^l$ *is a definable diffeomorphism, then*

$$\langle T, g \circ f, y\rangle = \varepsilon \langle T, f, g^{-1}(y)\rangle$$

whenever $y \in \mathbb{R}^l$ *with* $\dim(g \circ f)^{-1}(y) \cap \operatorname{spt} T \leq k - l$*,* $\dim(g \circ f)^{-1}(y) \cap \operatorname{spt} \partial T \leq k - l - 1$*. Here* $\varepsilon = 1$ *if* g *is orientation preserving, and* -1 *else.*

Note that the conditions on the dimensions are satisfied for almost all $y \in \mathbb{R}^l$. The above statement, but with the condition on the dimensions replaced by *for almost all* $y \in \mathbb{R}^l$, is well-known for any (not necessarily definable) normal current T (compare [**20**], 4.3). In the proof of existence and uniqueness of normal cycles, we will only use this weaker version of the above proposition. Only in the construction of the normal cycles associated to projections and convolutions of constructible functions, we will have to slice at special values and then we verify that the condition on the dimensions is satisfied.
For our purposes, the most important case is where the function f is the orthogonal projection π_W on a subspace W with $\dim W = \dim T$. In this case, we find a C^2-cell decomposition of $\operatorname{spt} T$ compatible with π_W. If D' is a cell of highest dimension in W, then $\pi_W^{-1}(D') \cap \operatorname{spt} T$ is a union of graphs on D'. It follows that for $y \in D'$, the intersection $\pi_W^{-1}(y) \cap \operatorname{spt} T$ is a finite union of points and the slice $\langle T, \pi_W, y\rangle$ (which is 0-dimensional) is the sum of the corresponding Dirac measures, counted with multiplicities according to the multiplicities of the cells of $\operatorname{spt} T$.

3. Support functions of Legendrian cycles

We fix the following notation. V will denote an oriented n-dimensional Euclidean vector space. The canonical projections are denoted by $\pi_1, \pi_2 : V \oplus V \to V$, $\tau_1, \tau_2 : V \to V \oplus V$ are the canonical embeddings. We define maps $m : \mathbb{R} \oplus V \oplus V \to V \oplus V, (\lambda, x, y) \mapsto (x, \lambda y)$ and $m_\lambda : V \oplus V \to V \oplus V, m_\lambda(x, y) := m(\lambda, x, y)$. Note that $m_0 = \tau_1 \circ \pi_1$. The scalar product is denoted by $u : V \oplus V \to \mathbb{R}, (x, y) \mapsto \langle x, y\rangle$.
We let $S(V)$ denote the unit sphere in V and $SV := V \times S(V) \subset V \oplus V$ the sphere bundle of V.
The canonical 1-form α on $V \oplus V$ is defined by $\alpha(v) = \langle y, (\pi_1)_* v\rangle$ for $v \in T_{(x,y)} V \oplus V$. We will not distinguish notationally between α and its restriction on SV, making the latter space into a contact manifold. After choice of an orthonormal basis of V, we get canonical coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ on $V \oplus V$ and α can be written as

$$\alpha = \sum_{i=1}^n y_i dx_i.$$

Definition 3.1. *A Legendrian current is a current* $T \in \mathcal{D}_{n-1}(V \oplus V)$ *such that* $\operatorname{spt} T$ *is contained in* SV *and such that*

$$T \llcorner \alpha = 0.$$

The 1-form α is a contact form on SV and $T\llcorner\alpha = 0$ for a definable current T is equivalent to saying that all $n-1$-cells in the support of T are horizontal, i.e. tangent to the contact distribution $\ker\alpha$. Note that $n-1$ is the critical dimension, i.e. there are no horizontal n-dimensional submanifolds in a $2n-1$-dimensional contact manifold.

DEFINITION 3.2. *We call a current $S \in \mathcal{D}_n(V \oplus V)$ conical if*

$$(m_\lambda)_* S = S$$

for $\lambda > 0$. S is called Lagrangian if, with $\omega := -d\alpha$ denoting the symplectic form on $V \oplus V$,

$$S\llcorner\omega = 0.$$

We recall that a linear subspace W of $V \oplus V$ is called *isotropic* if $\omega|_W = 0$. Then $\dim W \leq n$ and W is called *Lagrangian* if $\dim W = n$.

PROPOSITION 3.3.
a) *If T is a Legendrian cycle, then $T\llcorner\omega = 0$.*
b) *If S is a conical, definable Lagrangian current on $V \oplus V$, then $S\llcorner\alpha = 0$.*
c) *There is a one-to-one correspondence between compactly supported definable Legendrian cycles T and definable, conical Lagrangian cycles S on $V \oplus V$ such that $\pi_1(\operatorname{spt} S)$ is compact.*

PROOF. (Compare [**27**])

a) Let ϕ be an $n-3$-form on $V \oplus V$. Since $\partial T = T\llcorner\alpha = 0$, we obtain $T\llcorner\omega(\phi) = -T(d\alpha \wedge \phi) = T(\alpha \wedge d\phi) = 0$.
b) Fix coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ as above. If D is an n-dimensional cell in the support of S and $(x,y) \in D$, then $T_{(x,y)}D$ is Lagrangian. If $y = 0$, then α vanishes trivially on $T_{(x,y)}D$. If $y \neq 0$, then $v := \sum_{i=1} y_i \frac{\partial}{\partial y_i} \in T_{(x,y)}D$ (since S is conical) and $\alpha|_{T_{(x,y)}D} = -v \lrcorner \omega|_{T_{(x,y)}D} = 0$.
c) We write $[0,\infty)$ not only for the interval, but also for the 1-current defined by integration over it. Given a definable Legendrian cycle T,

$$\partial m_*([0,\infty) \times T) = -m_*(\delta_0 \times T) = -(\tau_1)_*(\pi_1)_* T.$$

The current $(\pi_1)_* T$ is a compactly supported definable $n-1$-cycle in V and can be filled by a compactly supported definable n-current U, i.e. $\partial U = (\pi_1)_* T$. Then

$$S := m_*([0,\infty) \times T) + (\tau_1)_* U$$

is a conical Lagrangian cycle and $\pi_1(\operatorname{spt} S) \subset \pi_1(\operatorname{spt} T) \cup \operatorname{spt} U$ is compact.

Now suppose S is a conical Lagrangian cycle with $\pi_1(\operatorname{spt} S)$ compact. We can assume that S is given by integration over oriented conical cells and define the current T by intersecting S with SV (i.e. by taking intersections of the conical cells of S

with SV, with the same multiplicities). Then T is a compactly supported Legendrian cycle.

It can be checked that the operations $T \mapsto S, S \mapsto T$ are inverse to each other, finishing the proof.

$\square$

DEFINITION 3.4. *Let $T \in \mathcal{D}_{n-1}(V \oplus V)$ be a compactly supported, definable Legendrian cycle and S the associated Lagrangian cycle. The support function of T is the (almost everywhere defined) function $h_T : V \mapsto \mathbb{Z}[R]$ with*

$$h_T(y) := u_*\langle S, \pi_2, y\rangle.$$

Since S is conical, the support function of a Legendrian cycle T is (almost everywhere) homogeneous in the sense of I.3.3, b).

4. Lipschitz continuity of support functions

THEOREM 4.1. *Let $T \in \mathcal{D}_{n-1}(V \oplus V)$ be a definable Legendrian cycle with compact support. Then h_T can be extended to a definable Lipschitz continuous function $V \to \mathbb{Z}[\mathbb{R}]$ (with respect to $\mathbf{F}$).*

PROOF. **Step 1:** Let S be the definable, conical Lagrangian cycle associated to T. Suppose that $\operatorname{spt} S \subset B(0, R) \times V$.
Fix a definable C^1-cell decomposition of $V \oplus V$, compatible with $\operatorname{spt} S$ and π_2 (compare Theorem I.1.4). By reverse induction we can also achieve that the boundary of a cell in V is a union of cells.
Fix an orthonormal base of V and let $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ denote canonical coordinates of $V \oplus V$.
Let $D \subset \operatorname{spt} S$ be an n-dimensional cell and $(x, y) \in D$. Given $v = \sum_{i=1}^n \left(c_i \frac{\partial}{\partial x_i} + d_i \frac{\partial}{\partial y_i}\right) \in T_{(x,y)}D$, the Legendrian condition implies that $\langle c, y\rangle = 0$. Therefore,

$$u_* v = \frac{d}{dt}\bigg|_{t=0} \langle x + tc, y + td\rangle = \langle x, d\rangle,$$

which implies that

$$(\pi_2, u)_* v = (d_1, \ldots, d_n, \langle x, d\rangle). \tag{1}$$

Suppose first that the rank of $\pi_2|_D$ is n. Then $D' := \pi_2(D) \subset V$ is an n-dimensional cell and D is the graph of a definable, C^1-smooth function $g : D' \to V$. With $f(y) := \langle g(y), y\rangle$ for $y \in D'$ we get $(\pi_2, u)(D) = \operatorname{graph} f$.
From Equation (1) we deduce that $\operatorname{grad} f(y) = g(y)$ for all $y \in D'$. Since $(g(y), y) \in \operatorname{spt} S \subset B(0, R) \times V$, the norm of the gradient of f is bounded by R, which implies that f is locally R-Lipschitz on D'.
If the rank of $\pi_2|_D$ is less than n, then Equation (1) implies that also the rank of $(\pi_2, u)|_D$ is less than n and thus $(\pi_2, u)_*[[D]] = 0$.

We obtain that $G := (\pi_2, u)_* S$ is given by integration over finitely many (say M) cells of $V \times \mathbb{R}$ which are graphs of locally R-Lipschitz functions on n-dimensional cells in V. In particular, G has no vertical components.
Note further that, with $\pi_z : V \times \mathbb{R} \to \mathbb{R}, (y, z) \mapsto z$ and $\pi_y : V \times \mathbb{R} \to V, (y, z) \mapsto y$, we get for almost all $y \in V$

$$\begin{aligned}
(\pi_z)_* \langle G, \pi_y, y \rangle &= (\pi_z)_* \langle (\pi_2, u)_* S, \pi_y, y \rangle \\
&= \underbrace{(\pi_z)_* \circ (\pi_2, u)_*}_{=u_*} \langle S, \underbrace{\pi_y \circ (\pi_2, u)}_{=\pi_2}, y \rangle \\
&= h(y),
\end{aligned}$$

i.e. G can be considered as “graph” of h.
Step 2: *Let y belong to an $n-1$-dimensional cell. Then h is continuous at y.*

LEMMA 4.2. *Let $D \subset V$ be a k-cell and let $f : D \to \mathbb{R}$ be a bounded and definable C^1-function. Then there exists a definable C^2-cell decomposition of ∂D such that for each cell D' of dimension $k-1$ there exists a unique continuous extension of f on $D \cup D'$.*

PROOF. This is a standard argument, a sketch of which will be given. We fix cell decompositions of the boundary of the graph of f (which is a bounded, definable, $k-1$-dimensional subset of $V \times \mathbb{R}$) and of the boundary of D which are compatible with the projection to V. Above an $k-1$-dimensional cell $D' \subset \partial D$, there can only be finitely many $k-1$-dimensional cells. Since D is locally connected, there is exactly one such cell and the result follows. ☐

By the Lemma, we find a refinement of the cell decomposition in such a way that each of the functions $f : D' \to \mathbb{R}$ can be continuously extended to $n-1$-cells in the boundary of D'.
Let D'' be a cell of V of dimension $n-1$. Let D'_1 and D'_2 be the two n-cells of V containing D'' in their boundary. Note that the induced orientations on D'' do not coincide.
By Step 1, there are representations of the form

$$\begin{aligned}
h(y) &= \sum_{i=1}^{k} a_i \delta_{f_i(y)} \quad \forall y \in D'_1 \\
h(y) &= \sum_{j=1}^{l} b_j \delta_{g_j(y)} \quad \forall y \in D'_2,
\end{aligned}$$

with locally R-Lipschitz continuous functions f and g.
By construction, the functions f_i (resp. g_j) extend by continuity to $D'_1 \cup D''$ (resp. $D'_2 \cup D''$). Let $r : D'' \to \mathbb{R}$ be the restriction of such a function to D''.

Let $I_r \subset \{1, \ldots, k\}$ be the set of indices i such that $f_i|_{D''} = r$ and $J_r \subset \{1, \ldots, l\}$ be the set of indices j such that $g_j|_{D''} = r$.
Since each i belongs to exactly one I_r and each j belongs to exactly one J_r, and since G has no vertical components, we get

$$
\begin{aligned}
G \llcorner \pi_y^{-1}(D'_1 \cup D'_2) = \sum_r \Bigg(& \sum_{i \in I_r} a_i [[\operatorname{graph} f_i : D'_1 \to \mathbb{R}]] + \\
& + \sum_{j \in J_r} b_j [[\operatorname{graph} g_j : D'_2 \to \mathbb{R}]] \Bigg)
\end{aligned}
$$

and

$$
\begin{aligned}
\partial G \llcorner D'' &= \sum_r \Bigg(\sum_{i \in I_r} a_i \partial [[\operatorname{graph} f_i : D'_1 \to \mathbb{R}]] + \\
&\quad + \sum_{j \in J_r} b_j \partial [[\operatorname{graph} g_j : D'_2 \to \mathbb{R}]] \Bigg) \\
&= \sum_r \Bigg(\sum_{i \in I_r} a_i [[\operatorname{graph} f_i : D'' \to \mathbb{R}]] - \\
&\quad - \sum_{j \in J_r} b_j [[\operatorname{graph} g_j : D'' \to \mathbb{R}]] \Bigg) \\
&= \sum_r \Bigg(\sum_{i \in I_r} a_i - \sum_{j \in J_r} b_j \Bigg) [[\operatorname{graph} r : D'' \to \mathbb{R}]].
\end{aligned}
$$

On the other hand, $\partial G = (\pi_2, u)_* \partial S = 0$ and thus $\sum_{i \in I_r} a_i = \sum_{j \in J_r} b_j$ for all r.
Let $y \in D''$ and $\epsilon > 0$. Since f_i and g_j can be continuously extended to D'', we get for all $y_1 \in D'_1$ and $y_2 \in D'_2$ sufficiently close to y that $|f_i(y_1) - f_i(y)| \leq \epsilon$ and $|g_j(y_2) - g_j(y)| \leq \epsilon$.
From

$$
\sum_{i \in I_r} a_i \delta_{f_i(y)} = \sum_{i \in I_r} a_i \delta_{r(y)} = \sum_{j \in J_r} b_j \delta_{r(y)} = \sum_{j \in J_r} b_j \delta_{g_j(y)}
$$

we deduce that

$$\begin{aligned}
\mathcal{F}(h(y_1) - h(y_2)) &= \mathcal{F}\left(\sum_r \left(\sum_{i\in I_r} a_i\delta_{f_i(y_1)} - \sum_{j\in J_r} b_j\delta_{g_j(y_2)}\right)\right) \\
&\leq \mathcal{F}\left(\sum_r \sum_{i\in I_r} a_i(\delta_{f_i(y_1)} - \delta_{f_i(y)})\right) + \\
&\quad + \mathcal{F}\left(\sum_r \sum_{j\in J_r} b_j(\delta_{g_j(y_2)} - \delta_{g_j(y)})\right) \\
&\leq \left(\sum_{i=1}^k |a_i| + \sum_{j=1}^l |b_j|\right)\epsilon.
\end{aligned}$$

This proves the claim.

Step 3:

Let $y_1, y_2 \in V$ be both contained in n-dimensional cells. For sufficiently small $\epsilon > 0$, each point y_1' with $\|y_1' - y_1\| \leq \epsilon$ satisfies $\mathcal{F}(h(y_1') - h(y_1)) \leq M\|y_1' - y_1\|$ and similarly for y_2' (see Step 1). With a random choice of $y_1' \in B(y_1, \epsilon)$ and $y_2' \in B(y_2, \epsilon)$, the line between y_1' and y_2' crosses finitely many $n-1$-dimensional cells and stays in the union of the n-dimensional cells otherwise.

By Step 1, the restriction of h to this line is locally MR-Lipschitz except for a finite number of points. In these points, h is continuous by Step 2. We deduce that h is MR-Lipschitz, and it follows that

$$\begin{aligned}
\mathcal{F}(h(y_1) - h(y_2)) &\leq \mathcal{F}(h(y_1) - h(y_1')) + \mathcal{F}(h(y_1') - h(y_2'))+ \\
&\quad + \mathcal{F}(h(y_2') - h(y_2)) \\
&\leq 2MR\epsilon + MR\|y_1' - y_2'\| \\
&\leq 4MR\epsilon + MR\|y_1 - y_2\|.
\end{aligned}$$

Since ϵ can be chosen arbitrary small, we obtain $\mathcal{F}(h(y_1) - h(y_2)) \leq MR\|y_1 - y_2\|$. Since the union of all n-dimensional cells is dense in V, h can be extended to an MR-Lipschitz continuous, definable function on V. $\square$

CHAPTER III

The normal cycle

As in previous chapters, V denotes an oriented, n-dimensional Euclidean vector space.

1. Construction of the normal cycle

Definition 1.1. *Let $\phi : V \to \mathbb{Z}$ be a constructible function with compact support. A compactly supported, definable Legendrian cycle $T \in \mathcal{D}_{n-1}(V \oplus V)$ is called normal cycle of ϕ if $h_T = h_\phi$ almost everywhere.*

Remark. The normal cycle of ϕ depends on the orientation of V. Indeed, changing the orientation of V does not alter h_ϕ, but $\langle T, \pi_2, y\rangle$ depends on the orientation of the target space V. Therefore, the normal cycle of ϕ with respect to the reversed orientation is minus the normal cycle of ϕ with respect to the given one.

Theorem 1.2. **Existence and uniqueness of the normal cycle** *Each compactly supported constructible function ϕ admits a unique normal cycle. Conversely, each compactly supported definable Legendrian cycle is the normal cycle of a unique constructible function with compact support.*

The first part of this theorem was discovered by Fu ([**26**]) using deep methods from Geometric Measure Theory. The proof we will give below only uses Lipschitz continuity of support functions and basic properties of constructible functions and definable currents.

Notation. The normal cycle of a compactly supported constructible function will be denoted by T_ϕ. The corresponding conical Lagrangian cycle will be denoted by S_ϕ. Given a compactly supported Legendrian cycle T (or a conical Lagrangian cycle S with $\pi_1(\operatorname{spt} S)$ compact), we denote by ϕ_T (or ϕ_S) the unique compactly supported constructible function with normal cycle T.

Proof. The proof of the second part is already contained in Chapter II. Indeed, if T is a compactly supported definable Legendrian cycle, then $h = h_T$ is definable, homogeneous and Lipschitz (Theorem II.4.1). Theorem I.4.2 implies that there is a unique constructible function ϕ with compact support such that $h_\phi = h$.

Conversely, let ϕ be constructible with compact support. By Theorem I.4.2, $h = h_\phi$ is definable, homogeneous and L-Lipschitz for some $L >$

1 (with respect to $\mathbf{F}$). We have to show that there exists a unique compactly supported, definable Legendrian cycle T with $h_T = h$.
Existence

LEMMA 1.3. *Let $D \subset V$ be a C^2-cell and $f \in C^2(D)$. Suppose that D is conical and f is homogeneous, i.e. $\lambda D = D$ and $f(\lambda y) = \lambda f(y)$ for all $\lambda > 0, y \in D$. Then*

$$\Gamma(D, f) := \{(x, y) \in V \oplus V : y \in D, \langle x, v\rangle = v(f) \forall v \in T_y D\}$$

is a conical Lagrangian submanifold of $V \oplus V$.

PROOF. Since D is conical, $y \in T_y D$ for all $y \in D$. By homogeneity of f, $\langle x, y\rangle = f(y)$ for all $(x, y) \in \Gamma(D, f)$.
Let $(x(t), y(t))$ be a differentiable curve in $\Gamma(D, f)$ with $(x(0), y(0)) = (x, y)$. Then $v := y'(0) \in T_y D$. We obtain

$$\left.\frac{d}{dt}\right|_{t=0} \langle x(t), y(t)\rangle = \left.\frac{d}{dt}\right|_{t=0} f(y(t)) = v(f).$$

On the other hand,

$$\left.\frac{d}{dt}\right|_{t=0} \langle x(t), y(t)\rangle = \langle x'(0), y\rangle + \underbrace{\langle x, y'(0)\rangle}_{=v(f)}.$$

Comparing these formulas yields $\langle x'(0), y\rangle = 0$, which shows that $\alpha|_{\Gamma(D,f)} = 0$. Differentiation yields that $d\alpha|_{\Gamma(D,f)} = 0$, i.e. $\Gamma(D, f)$ is Lagrangian.
It is clear that $\Gamma(D, f)$ is conical. $\square$

LEMMA 1.4. *Let D, f be as in the preceding lemma and suppose that $\|\operatorname{grad} f\| \leq L$. Let S be a definable, conical, $n-1$-dimensional current on $V \oplus V$ with $\operatorname{spt} S \subset \overline{\Gamma(D, f)}$, $\operatorname{spt} \partial S \subset \partial\Gamma(D, f)$ and $\pi_1(\operatorname{spt} S)$ compact.*
If $\dim D < n - 1$ or $\dim D = n - 1$ and $(\pi_2)_* S = 0$, then there exists a conical, definable n-current S' with $\operatorname{spt} S' \subset \overline{\Gamma(D, f)}$, $\operatorname{spt}(\partial S - S') \subset \partial\Gamma(D, f)$ and such that $\pi_1(\operatorname{spt} S')$ is contained in the convex hull of $\pi_1(\operatorname{spt} S) \cup B(0, L)$.

PROOF. For $y \in D$, set $g(y) := \sum_{i=1}^{\dim D} e_i(f) e_i$, where $e_1, \ldots, e_{\dim D}$ is an orthonormal base of $T_y D$ (if $\dim D = 0$, set $g(y) = 0$). Clearly $(g(y), y) \in \Gamma(D, f)$ and $\|g(y)\| \leq L$.
Define a homotopy

$$H : [0, 1] \times \Gamma(D, f) \to \Gamma(D, f), (t, (x, y)) \mapsto (tx + (1 - t)g(y), y)$$

and set $S' := H_*([0, 1] \times S)$.
By the homotopy formula II.1.3, up to a current with support in $\partial\Gamma(D, f)$,

$$\partial S' = \underbrace{H_*(\delta_1 \times S)}_{=S} - H_*(\delta_0 \times S).$$

If $\dim D < n-1$, then the second term vanishes since it is an $n-1$-current supported in the $\dim D$-dimensional set $\{(g(y), y) : y \in D\}$.
If $\dim D = n-1$ and $(\pi_2)_* S = 0$, then

$$H_*(\delta_0 \times S) = (H_0)_* S = (g, \mathrm{id})_* \circ (\pi_2)_* S = 0.$$

In both cases, $\pi_1(\operatorname{spt} S')$ is contained in the convex hull of $\pi_1(\operatorname{spt} S) \cup B(0, L)$. $\square$

LEMMA 1.5. *Let $h : V \to \mathbb{Z}[\mathbb{R}]$ be homogeneous, definable and L-Lipschitz. Then there exist finite C^2-cell decompositions of $V \oplus V$ and V, compatible with π_2, such that*

a) *each cell $D \subset V$ is conical and each cell $\tilde{D} \subset V \oplus V$ is conical in the second coordinate;*
b) *for each cell $D \subset V$, there exists a finite family $F(D)$ of definable C^2-functions $f_1 < f_2 < \ldots < f_k$ and integers $a_1, \ldots, a_k$ with*
$$h(y) = \sum_{i=1}^{k} a_i \delta_{f_i(y)} \forall y \in D;$$
c) *if $D' \subset \partial D, \dim D' = \dim D - 1$ and $f \in F(D)$, then there exists $f' \in F(D')$ which is the restriction of the continuous extension of f to D';*
d) *the boundary of each cell is a union of cells;*
e) *if $D \subset V$ is a cell and $f \in F(D)$, then $\Gamma(D, f)$ is a union of cells.*

PROOF. In the first step, we construct a cell decomposition of V with a)-d). Since h is homogeneous and definable, we find a cell decomposition of V such that h is given above each cell by $h(y) = \sum_{i=1}^k a_i \delta_{f_i(y)}$. For n-dimensional cells D, we set $F(D) = (f_1, \ldots, f_k)$. Using Lemma II.4.2, we can subdivide the $n-1$-skeleton in such a way that all functions $f \in F(D)$ can be continuously extended to cells of dimension $n-1$. For a cell D' of dimension $n-1$, we let $F(D')$ be the set of restrictions of all functions belonging to some $F(D)$ with $\dim D = n$ and $D' \subset \partial D$.
Subdividing the $n-2$-skeleton, we can assume that all functions $f \in F(D), \dim D = n-1$, extend continuously to $n-2$-cells. We define $F(D')$ for $n-2$-cells similarly as above and continue in this way. After n steps, we get a cell decomposition of V with a)-d).
Note that any subdivision of this cell decomposition also satisfies a)-d) (we let $F(D')$ be the set of restrictions of functions from $F(D)$, where D is the unique cell of the original decomposition containing D').
In the second step, we construct a cell decomposition of $V \oplus V$ which is π_2-compatible with some subdivision of the given cell decomposition and which satisfies d) and e). We choose a cell decomposition of $V \oplus V$ such that $\Gamma(D, f)$ is a union of cells for each D of dimension n and $f \in F(D)$. By subdividing, we can achieve that the boundary of

each cell is a cell. By subdividing again, we achieve that the sets $\Gamma(D,f), f \in F(D)$ with $\dim D = n-1$ are unions of cells. Continuing in this way, we obtain π_2-compatible cell decompositions with a)-e). $\square$

We fix cell decompositions as in Lemma 1.5 and set

$$\mathcal{Y}_{\leq k} := \bigcup_{\dim D \leq k, f \in F(D)} \Gamma(D,f).$$

LEMMA 1.6. *Let $D \subset V$ be a cell and $f \in F(D)$. Then*

$$\partial\Gamma(D,f) \subset \mathcal{Y}_{\leq \dim D-1} \cup M,$$

where M is a subset of dimension $< n-1$.

PROOF. Let $\tilde{D}_1 \subset \partial\Gamma(D,f)$ be an $n-1$-cell and $D_1 := \pi_2(\tilde{D}_1)$. Then $D_1 \subset \partial D$, in particular $\dim D_1 \leq \dim D - 1$. By Lemma 1.3 and Stokes's theorem, α vanishes on $\tilde{D}_1$.
Let $(x,y) \in \tilde{D}_1$. Then there exists a sequence $(x_i,y_i) \in \Gamma(D,f)$ converging to (x,y). As was remarked above, $\langle x_i, y_i\rangle = f(y_i)$. By continuity, $\langle x,y\rangle = f(y)$.
Consider a differentiable curve $\gamma(t) = (x(t),y(t))$ in $\tilde{D}_1$ with $(x,y) = (x(0),y(0))$ and set $v := y'(0)$. Then

$$v(f) = \frac{d}{dt}\bigg|_{t=0} f(y(t)) = \frac{d}{dt}\bigg|_{t=0} \langle x(t),y(t)\rangle = \langle x,v\rangle + \underbrace{\langle x'(0),y(0)\rangle}_{=\alpha(\gamma'(0))=0}.$$

Since $\pi_2 : \tilde{D}_1 \to D_1$ is submersive, it follows that $(x,y) \in \Gamma(D_1,f)$. $\square$

Now we can complete the construction of the normal cycle.
We define a sequence of currents $S_k, k = 0,1,\ldots,n$ such that

- S_k is a conical, definable Lagrangian current with $\pi_1(\operatorname{spt} S_k) \subset B(0,L)$;
- $u_*\langle S_k, \pi_2, y\rangle = h(y)$ for almost all $y \in V$;
- $\operatorname{spt} \partial S_k \subset \overline{\mathcal{Y}_{\leq n-k-1}}$.

For a cell D of dimension n, $h|_D$ is given as a finite combination

$$h|_D = \sum_{i=1}^{k(D)} a_i^D \delta_{f_i^D},$$

with functions $f_1^D < \ldots < f_{k(D)}^D$ from $F(D)$.
We set

$$S_0 := \sum_{\dim D=n} \sum_{i=1}^{k(D)} a_i^D [[\Gamma(D,f_i^D)]].$$

From Lemma 1.3 we deduce that S_0 is a definable, conical, Lagrangian current.

Moreover, $\pi_1(\operatorname{spt} S_0) \subset B(0,L)$ by Lemma I.4.5 and

$$\begin{aligned} u_*\langle S_0, \pi_2, y\rangle &= \sum_{i=1}^{k(D)} a_i^D u_*\langle [[\Gamma(D, f_i^D)]], \pi_2, y\rangle \\ &= \sum_{i=1}^{k(D)} a_i^D \delta_{\langle \operatorname{grad} f_i^D(y), y\rangle} \\ &= \sum_{i=1}^{k(D)} a_i^D \delta_{f_i^D(y)} \\ &= h(y) \end{aligned}$$

for $y \in D, \dim D = n$. This means that $u_*\langle S_0, \pi_2, y\rangle = h(y)$ for almost all $y \in V$.

From Lemma 1.6 we see that ∂S_0 is a cycle supported in $\overline{\mathcal{Y}}_{\leq n-1}$.

Let D be an $n-1$-cell, $f \in F(D)$ and D_1, D_2 be the n-cells neighboring D. Then

$$(\partial[[D_1]] + \partial[[D_2]]) \llcorner D = 0.$$

Suppose $h(y) = \sum_{i=1}^k a_i \delta_{f_i}$ on D_1. Let s_1 be the sum of those a_i for which $f_i|_D = f$. We define s_2 in a similar way. From the continuity of the support function, we obtain $s_1 = s_2$. Indeed, by b) and c), both s_1 and s_2 equal the coefficient of δ_f in $h|D$.

For each function f_i with $f_i|_D = f$ (and only for those), we get as in the proof of Lemma 1.6 that

$$\partial \Gamma(D_1, f_i) \cap \pi_2^{-1} D \subset \Gamma(D, f)$$

and

$$\begin{aligned} (\pi_2)_*(\partial[[\Gamma(D_1, f_i)]] \llcorner \pi_2^{-1} D) &= (\pi_2)_* \partial[[\Gamma(D_1, f_i)]] \llcorner D \\ &= \partial[[D_1]] \llcorner D. \end{aligned}$$

In the same way, if $h(y) = \sum_{i=1}^{k'} a_i' \delta_{f_i'}$ on D_2 and if $f_i'|_D = f$, then

$$(\pi_2)_*(\partial[[\Gamma(D_2, f_i')]] \llcorner \pi_2^{-1} D) = \partial[[D_2]] \llcorner D.$$

We deduce that

$$(\pi_2)_*(\partial S_0 \llcorner \Gamma(D, f)) = s_1 \partial[[D_1]] \llcorner D + s_2 \partial[[D_2]] \llcorner D = 0.$$

We apply Lemma 1.4 to the currents $(\partial S_0) \llcorner \Gamma(D, f)$ (where D runs over all $n-1$-dimensional cells and $f \in F(D)$) to deduce that there exists a conical, definable, Lagrangian current S_0' with $\pi_1(\operatorname{spt} S_0') \subset B(0,L)$, $\operatorname{spt} S_0' \subset \overline{\mathcal{Y}}_{n-1}$ and $\operatorname{spt}(\partial S_0 - \partial S_0') \subset \overline{\mathcal{Y}}_{\leq n-2}$. Hence $S_1 := S_0 - S_0'$ satisfies all conditions.

Suppose $S_k, 0 < k < n$ is already defined. Then ∂S_k is an $n-1$ cycle with support in $\overline{\mathcal{Y}}_{\leq n-k-1}$. Applying Lemma 1.4 yields a conical, definable, Lagrangian current S_k' with $\pi_1(\operatorname{spt} S_k') \subset B(0,L)$, $\operatorname{spt} S_k' \subset$

$\overline{\mathcal{Y}}_{n-k-1}$ and $\operatorname{spt}(\partial S_k - \partial S_k') \subset \overline{\mathcal{Y}}_{\leq n-k-2}$. Hence $S_{k+1} := S_k - S_k'$ satisfies all conditions.

In particular, $S := S_n$ is a conical, definable Lagrangian cycle such that $h_S(y) = h(y)$ for almost all $y \in V$ and $\pi_1(\operatorname{spt} S) \subset B(0, L)$. Let T be the associated Legendrian cycle. Then T is compactly supported, definable and $h_T(y) = h_S(y) = h(y)$ for almost all $y \in V$.

Uniqueness

It suffices to show that $h_S = 0$ implies $S = 0$ for compactly supported, definable conical Lagrangian cycles S.

Claim: $h_S = 0$ implies that $\langle S, \pi_2, y\rangle = 0$ for almost all $y \in V$.

To prove the claim, we fix C^2-cell decompositions of $V \oplus V$ and V which are compatible with π_2 and $\operatorname{spt} S$. If the conclusion does not hold, there exist a cell D of V of dimension n, finitely many pairwise and pointwise different definable C^2 functions $f_1, \ldots, f_k : D \to V$ and non-vanishing natural numbers such that

$$\langle S, \pi_2, y\rangle = \sum_{i=1}^{k} a_i \delta_{(f_i(y), y)} \quad \forall y \in D.$$

For almost all $y \in D$ we have $h_S(y) = \sum_{i=1}^k a_i \delta_{\langle f_i(y), y\rangle} = 0$. This implies that $k > 1$ and that there exists some index $i \neq 1$ with $\langle f_i(y), y\rangle = \langle f_1(y), y\rangle$. We thus find an open subset D' of D and an index $i \neq 1$, such that $\langle f_i(y), y\rangle = \langle f_1(y), y\rangle$ for all $y \in D'$.

Let $y \in D'$. The Legendrian condition implies that $\langle df_1(v), y\rangle = \langle df_i(v), y\rangle = 0$ for all $v \in T_y D'$. Setting $v := f_1(y) - f_i(y) \neq 0$ we obtain that

$$\underbrace{\left.\frac{d}{dt}\right|_{t=0} \langle f_1(y+tv) - f_i(y+tv), y+tv\rangle}_{=0 \text{ since } y+tv \in D' \text{ for small } t} = \underbrace{\langle f_1(y) - f_i(y), v\rangle}_{=\langle v,v\rangle \neq 0}.$$

This is a contradiction and finishes the proof of the claim.

Let m be the dimension of the projection $\pi_2(\operatorname{spt} S)$. Then $m < n$, since $\langle S, \pi_2, y\rangle = 0$ for almost all $y \in V$. We may choose coordinates in such a way that $\dim \psi \circ \pi_2(\operatorname{spt} S) = m$, where $\psi : V \to \mathbb{R}^m$ denotes projection on the first m coordinates.

Suppose $S \neq 0$. Fix compatible C^2-cell decompositions of $\operatorname{spt} S$, $\pi_2(\operatorname{spt} S)$ and $\psi \circ \pi_2(\operatorname{spt} S)$. Let D' be an m-dimensional cell of $\psi \circ \pi_2(\operatorname{spt} S)$. A cell $D \subset \pi_2(\operatorname{spt} S)$ with $\psi(D) = D'$ is a graph, since bands would have dimensions strictly larger than m. It follows that

$$A := \psi^{-1}(y') \cap \pi_2(\operatorname{spt} S) \subset V$$

is finite for almost all $y' \in D'$.

The slice $\langle S, \psi \circ \pi_2, y'\rangle$ is a non-vanishing definable cycle with support in $V \times A$. For some $y \in A$, its restriction R to $V \times \{y\}$ is again a non-vanishing definable cycle. Let D be the cell containing y.

The cell decomposition of $\operatorname{spt} S$ induces a natural cell decomposition of $\operatorname{spt} R$, with cells being the intersections $\tilde{D}_y := \tilde{D} \cap (V \times \{y\})$, where $\tilde{D}$ runs over all cells of $\operatorname{spt} S$.
Let $\tilde{D} \subset \operatorname{spt} S$ be an n-dimensional cell with $\tilde{D}_y \neq \emptyset$. Then $D = \pi_2(\tilde{D})$. Let $v \in T_{(x,y)}\tilde{D}_y$ be a tangent vector. Since $\tilde{D}$ is Lagrangian and $T_{(x,y)}\tilde{D}_y \subset T_{(x,y)}\tilde{D}$, it follows that

$$\langle d\pi_1(v), d\pi_2(w)\rangle = \langle d\pi_1(v), d\pi_2(w)\rangle - \langle \underbrace{d\pi_2(v)}_{=0}, d\pi_1(w)\rangle = \omega(v,w) = 0$$

for all $w \in T_{(x,y)}\tilde{D}$. In other words, $d\pi_1(v)$ is orthogonal to T_yD.
Let $\phi : V \to T_yD$ denote orthogonal projection. Then the rank of $\phi \circ \pi_1$, restricted to $\tilde{D}_y$, is 0, which implies that there exists a finite set $B \subset T_yD$ with

$$\tilde{D}_y \subset \phi^{-1}B \times \{y\}.$$

Since this is true for all $\tilde{D}$ as above (where B may differ), R is a non-vanishing definable $n-m$-cycle with support contained in a finite disjoint union of $n-m$-dimensional affine subspaces. This contradicts the constancy theorem II.1.4. $\square$

2. Properties of the normal cycle

2.1. Projections. Let $W \subset V$ be an oriented linear subspace of dimension l, $W^\perp$ its orthogonal complement, oriented in such a way that $W^\perp \oplus W$ has the same orientation as V, and let $\pi_W : V \to W$ and $\pi_{W^\perp} : V \to W^\perp$ be the orthogonal projections.

Proposition 2.1. *Let ϕ be a compactly supported constructible function on V. Then*

$$S_{(\pi_W)_*\phi} = \underbrace{\langle (\pi_W, \mathrm{id})_* S_\phi, \pi_{W^\perp} \circ \pi_2, 0\rangle}_{=:\pi_W(S_\phi)}. \tag{2}$$

The slice on the right hand exists, is supported in $W \oplus W$ and can be considered as a current on $W \oplus W$.

Proof. *Claim 1: The slice exists.*
Let $S := S_\phi$ and

$$A := \{(x,y) \in \operatorname{spt} S : y \in W\}.$$

Let $w_1', \ldots, w_{n-l}'$ denote an orthogonal base of $W^\perp$.
Fix a C^2-cell decomposition of A, a cell D and $(x,y) \in D$. Suppose that $\dim D = d$ and that the vectors $(v_i, w_i) \in T_{(x,y)}D, i = 1, \ldots, d$ form a base of $T_{(x,y)}D$. Since S_ϕ is Lagrangian, $\omega((v_i,w_i),(v_j,w_j)) = 0$. From $w_i \in W$ we infer that $\langle v_j, w_i\rangle = \langle \pi_W(v_j), w_i\rangle$.
Let L be the subspace generated by the vectors $(\pi_W(v_i), w_i), i = 1, \ldots, d$. The subspaces L and $(\{0\} \times W^\perp)$ are transversal and their sum is an

isotropic subspace of $V \oplus V$, hence of dimension $\leq n$. It follows that

$$\dim L + \underbrace{\dim(\{0\} \times W^\perp)}_{=n-l} \leq n.$$

We deduce that $rank(\pi_W, \mathrm{id})|_D \leq l$ and thus $\dim(\pi_W, \mathrm{id})(D) \leq l$. Since $(\pi_W, \mathrm{id})(A)$ is a union of such sets, it has dimension $\leq l$. It follows that

$$\dim \big((\mathrm{spt}(\pi_W, \mathrm{id})_* S) \cap (\pi_{W^\perp} \circ \pi_2)^{-1}(0)\big) \leq l, \tag{3}$$

which implies that the slice on the right hand side of (2) exists.
Claim 2: $\pi_W(S)$ is a definable conical Lagrangian cycle in $W \oplus W$.
From Proposition II.2.1 b) we see that the right hand side of (2) is a definable cycle.
With the notations of Section II.3 and using II.2.1, c) we see that

$$\begin{aligned}
(m_\lambda)_* \langle (\pi_W, \mathrm{id})_* S_\phi, \pi_{W^\perp} \circ \pi_2, 0\rangle &= (m_\lambda)_* \langle (\pi_W, \mathrm{id})_* S_\phi, \pi_{W^\perp} \circ \pi_2 \circ m_\lambda, 0\rangle \\
&= \langle \underbrace{(m_\lambda)_* \circ (\pi_W, \mathrm{id})_*}_{=(\pi_W, \mathrm{id})_* \circ (m_\lambda)_*} S, \pi_{W^\perp} \circ \pi_2, 0\rangle \\
&= \langle (\pi_W, \mathrm{id})_* S, \pi_{W^\perp} \circ \pi_2, 0\rangle.
\end{aligned}$$

Hence $\pi_W(S)$ is conical.
Since the support of $\pi_W(S)$ is contained in $(\pi_W, \mathrm{id})(A)$, the proof of Claim 1 also shows that this current is Lagrangian.
Claim 3: The support function of $\pi_W(S)$ equals $h_S|_W$.
We want to apply Proposition II.2.1 d) to the current $(\pi_W, \mathrm{id})_* S$ and the orthogonal projections onto the spaces $V \oplus W$ and $V \oplus W^\perp$. We have to check the condition on the dimension. Since $\partial(\pi_W, \mathrm{id})_* S = \partial S = 0$, there are only three conditions. The first one is already proved, see inequality (3).
Since $\mathrm{spt}(\pi_W, \mathrm{id})_* S$ is a definable set of dimension $\leq n$, we get for almost all $y \in W$

$$\dim(\mathrm{spt}(\pi_W, \mathrm{id})_* S \cap (\pi_W \circ \pi_2)^{-1}(y)) \leq n - l.$$

Inequality (3) also implies that for almost all $y \in W$

$$\dim(\mathrm{spt}(\pi_W, \mathrm{id})_* S \cap (\pi_{W^\perp} \circ \pi_2)^{-1}(0) \cap (\pi_W \circ \pi_2)^{-1}(y)) \leq 0.$$

We can therefore apply Proposition II.2.1 d) to conclude that

$$\begin{aligned}
&\langle \langle (\pi_W, \mathrm{id})_* S, \pi_{W^\perp} \circ \pi_2, 0\rangle, \pi_W \circ \pi_2, y\rangle \\
&\quad = \langle (\pi_W, \mathrm{id})_* S, \underbrace{(\pi_{W^\perp} \circ \pi_2, \pi_W \circ \pi_2)}_{=\pi_2}, (0, y)\rangle = (\pi_W, \mathrm{id})_* \langle S, \pi_2, y\rangle
\end{aligned}$$

for almost all $y \in W$.
From $u \circ (\pi_W, \mathrm{id}) = u$ on $V \oplus W$ we obtain that the support function of the cycle $\langle (\pi_W, \mathrm{id})_* S, \pi_{W^\perp} \circ \pi_2, 0\rangle$ equals the support function of S for almost all $y \in W$. Since both functions are Lipschitz continuous (Theorem II.4.1), they coincide for all $y \in W$. $\square$

2.2. Products.

PROPOSITION 2.2. *Given oriented Euclidean vector spaces V and W and compactly supported constructible functions ϕ on V and ψ on W,*

$$S_{\phi\otimes\psi} = S_\phi \times S_\psi.$$

PROOF. It is easily checked that $S_\phi \times S_\psi$ is again a definable, conical Lagrangian cycle.
With $m : \mathbb{R}\times\mathbb{R} \to \mathbb{R}, (y_1, y_2) \mapsto y_1 + y_2$, and using Proposition I.3.4, we get for almost all $y_1 \in V, y_2 \in W$

$$\begin{aligned}
h_{S_\phi\times S_\psi}(y_1, y_2) &= u_*\langle S_\phi \times S_\psi, \pi_2, (y_1, y_2)\rangle \\
&= u_* \left(\langle S_\phi, \pi_2, y_1\rangle \times \langle S_\psi, \pi_2, y_2\rangle\right) \\
&= m_*(h_{S_\phi}(y_1) \times h_{S_\psi}(y_2)) \\
&= h_{S_\psi}(y_1) \cdot h_{S_\psi}(y_2) \\
&= h_\phi(y_1) \cdot h_\psi(y_2) \\
&= h_{\phi\otimes\psi}(y_1, y_2).
\end{aligned}$$

□

2.3. Linear transformations and convolution.

PROPOSITION 2.3. *Let $A \in GL(V)$ and ϕ a compactly supported constructible function. Then*

$$S_{A_*\phi} = \operatorname{sgn}(\det A) \cdot (A, (A^*)^{-1})_* S_\phi.$$

PROOF. Let $\epsilon := \operatorname{sgn}(\det A)$. It can be checked that $\epsilon(A, (A^*)^{-1})_* S$ is again a definable, conical Lagrangian cycle. Then the assertion follows from

$$\begin{aligned}
h_{\epsilon(A,(A^*)^{-1})_* S_\phi}(y) &= \epsilon u_* \langle (A, (A^*)^{-1})_* S_\phi, \pi_2, y\rangle \\
&= \epsilon \underbrace{u_* \circ (A, (A^*)^{-1})_*}_{=u_*} \langle S_\phi, \underbrace{\pi_2 \circ (A, (A^*)^{-1})}_{=(A^*)^{-1}\circ\pi_2}, y\rangle \\
&= u_* \langle S_\phi, \pi_2, A^* y\rangle \\
&= h_\phi(A^* y)
\end{aligned}$$

and Proposition I.3.4. □

PROPOSITION 2.4. a) *Let ϕ, ψ be compactly supported constructible functions on V. Let $\Delta \subset V \oplus V$ be the diagonal and $\tau : \Delta \to V, (x, x) \mapsto x$. Then the normal cycle of $\phi * \psi$ is given by*

$$S_{\phi*\psi} = (2\tau, \tau)_* \pi_\Delta (S_\phi \times S_\psi). \quad (4)$$

b) *Let $\psi = 1_{B(0,\epsilon)}$ and* $\exp^\epsilon : SV \to SV, (x, v) \mapsto (x + \epsilon v, v)$. *Then*

$$T_{\phi*\psi} = \exp^\epsilon_* T_\phi.$$

PROOF. a) We do not prove that the current on the right hand side of (4) is a definable, conic Lagrangian cycle, this is a straightforward computation. Now note that the following diagram commutes:

$$\begin{array}{ccc} \Delta \times \Delta & \stackrel{(2\tau,\tau)}{\rightarrow} & V \oplus V \\ \pi_2 \downarrow & & \pi_2 \downarrow \\ \Delta & \stackrel{\tau}{\rightarrow} & V \end{array}$$

With $u_\Delta : \Delta \times \Delta \to \mathbb{R}, ((x,x),(y,y)) \mapsto 2\langle x, y\rangle$ denoting the restriction of the scalar product of $V \oplus V$ to Δ, we also have $u \circ (2\tau, \tau) = u_\Delta$.

By Propositions II.2.1, 2.1 and 2.2, we have for almost all $y \in V$

$$\begin{aligned} h_{(2\tau,\tau)_*\pi_\Delta(S_\phi \times S_\psi)}(y) &= u_*\langle (2\tau,\tau)_*\pi_\Delta(S_\phi \times S_\psi), \pi_2, y\rangle \\ &= u_* \circ (2\tau,\tau)_*\langle \pi_\Delta(S_\phi \times S_\psi), \underbrace{\pi_2 \circ (2\tau,\tau)}_{=\tau\circ\pi_2}, y\rangle \\ &= (u_\Delta)_*\langle \pi_\Delta(S_\phi \times S_\psi), \pi_2, \underbrace{\tau^{-1}(y)}_{=(y,y)}\rangle \\ &= h_{\pi_\Delta(S_\phi \times S_\psi)}(y,y) \\ &= h_\phi(y) \cdot h_\psi(y) \\ &= h_{\phi*\psi}(y). \end{aligned}$$

b) It is easily checked that $\exp_*^\epsilon T_\phi$ is again a definable Legendrian cycle. Its support is contained in the ϵ-neighborhood of the support of T_ϕ, and thus compact. Now for almost all $v \in S(V)$

$$\begin{aligned} h_{\exp_*^\epsilon T_\phi}(v) &= u_*\langle \exp_*^\epsilon T_\phi, \pi_2, v\rangle \\ &= u_* \exp_*^\epsilon \langle T_\phi, \underbrace{\pi_2 \circ \exp^\epsilon}_{=\pi_2}, v\rangle \\ &= \delta_\epsilon \cdot u_*\langle T_\phi, \pi_2, v\rangle \\ &= h_\psi(v) \cdot h_\phi(v) \\ &= h_{\phi*\psi}(v), \end{aligned}$$

which shows that $\exp_*^\epsilon T_\phi$ is the normal cycle of $\phi * \psi$.

□

COROLLARY 2.5. *Let $v \in V$ and denote by $\mathrm{tr}_v : V \to V, x \mapsto x + v$ the translation. For a constructible function ϕ with compact support,*

$$S_{\phi\circ\mathrm{tr}_v} = (\mathrm{tr}_{-v}, \mathrm{id})_* S_\phi.$$

3. Support of the normal cycle

THEOREM 3.1. *Let $\phi : V \to \mathbb{Z}$ be a constructible function with compact support and $T := T_\phi$, $S := S_\phi$. Then* a)

$$\pi_1(\operatorname{spt} T) \subset \{x \in V : \phi \textit{ not constant near } x\}$$
$$\pi_1(\operatorname{spt} S) \subset \operatorname{spt} \phi;$$

b) *if ϕ is constant near x, then*

$$\phi(x) = [\pi_1(T)] \in H_{n-1}(V, V \setminus \{x\}) = \mathbb{Z};$$

c) *there exists a C^2-cell decomposition of* $\operatorname{spt}\phi$ *such that*

$$\operatorname{spt} S_\phi \subset \bigcup_{D \textit{ cell}} \operatorname{Nor} D.$$

Here $\operatorname{Nor} D = \{(x,y) \in V \oplus V : x \in D, y \perp T_x D\}$ *denotes the normal space of a cell D.*

PROOF. a) Suppose first that $\operatorname{spt}\phi \subset B(0,R)$. By the Remark just before the uniqueness proof of Theorem I.4.2, $h = h_\phi$ is given above each cell by functions whose gradients are bounded by $6R$. The construction in the proof of Theorem 1.2 can therefore be carried out with $L := 6R$ and shows that $\pi_1(\operatorname{spt} T_\phi) \subset B(0,6R)$ and $\pi_1(S_\phi) \subset B(0,6R)$

Corollary 2.5 implies that, whenever $\operatorname{spt}\phi \subset B(x,R)$ with $x \in V, R > 0$, then $\pi_1(\operatorname{spt} T) \subset B(x,6R)$ and $\pi_1(\operatorname{spt} S_\phi) \subset B(x,6R)$.

Now let ϕ be constant, say a, near $x \in V$. Then there exists $\epsilon > 0$ such that $\phi(y) = a$ for $y \in B(x,\epsilon)$. Set $\phi_0 := a 1_{B(x,\epsilon)}$ and let $T_0 := T_{\phi_0}$, $S_0 := S_{\phi_0}$.

Since $x \notin \operatorname{spt}(\phi - \phi_0)$, we can use compactness to write $\phi - \phi_0$ as a finite sum $\phi - \phi_0 = \sum_{i=1}^k \phi_i$ such that $\operatorname{spt}\phi_i \subset \operatorname{spt}(\phi - \phi_0)$ and such that $\operatorname{spt}\phi_i$ is contained in some ball $B(x_i, r_i)$ with the property that $x \notin B(x_i, 6r_i)$.

Let $T_i := T_{\phi_i}, S_i = S_{\phi_i}$. Then $\pi_1(\operatorname{spt} T_i) \subset B(x_i, 6r_i)$ and $\pi_1(\operatorname{spt} S_i) \subset B(x_i, 6r_i)$, i.e. $x \notin \pi_1(\operatorname{spt} T_i)$ and $x \notin \pi_1(\operatorname{spt} S_i)$.

Since $T - T_0 = \sum_{i=1}^k T_i$, $x \notin \pi_1(\operatorname{spt}(T - T_0))$. An easy computation shows that, in the case $a \neq 0$, $\pi_1(\operatorname{spt} T_0) = S(x,\epsilon)$. Therefore we obtain $x \notin \pi_1(\operatorname{spt} T)$.

If $x \notin \operatorname{spt}\phi$, then $S_0 = 0$. Thus $S = \sum_{i=1}^k S_i$ and we deduce that $x \notin \pi_1(\operatorname{spt} S)$.

b) Note that $\operatorname{spt}(\pi_1(T_i)) \subset \pi_1(\operatorname{spt} T_i)$ is supported in a ball not containing x, hence $[\pi_1(T_i)] = 0$. Therefore

$$[\pi_1(T)] = [a\pi_1(T_0)] = a[S^{n-1}(x,\epsilon)] = \phi(x).$$

c) By a), we find definable C^2-cell decompositions of $\operatorname{spt} S$ and $\operatorname{spt}\phi$, compatible with π_1. We can suppose that all cells of $\operatorname{spt} S$ are conical. Let D' be such a cell, $(x,y) \in D'$ and $D := \pi_1(D')$. Then there are finitely many vectors $v_1, \ldots, v_d \in$

$T_{(x,y)}D'$ such that $d\pi_1(v_i), i = 1, \ldots, d$ span T_xD. Now $0 = \alpha(v_i) = \langle y, d\pi_1(v_i)\rangle$, which implies that $y \perp T_xD$.

□

THEOREM 3.2. *Let $T = T_\phi$ be the normal cycle of the compactly supported constructible function ϕ. Let $T_\epsilon := \exp^\epsilon_* T$ be the image of T under the geodesic flow of SV after time $\epsilon > 0$. Then for every $x \in V$*

$$\phi(x) = \lim_{\epsilon \to 0^+} [(\pi_1)_* T_\epsilon] \in H_{n-1}(V, V \setminus \{x\}).$$

PROOF. By Proposition 2.4 b), T_ϵ is the normal cycle of the convolution $\phi_\epsilon := \phi * 1_{B(0,\epsilon)}$. For all $z \in V$

$$\phi_\epsilon(z) = \int_V \phi(y) \underbrace{1_{B(0,\epsilon)}(z-y)}_{=1_{B(z,\epsilon)}(y)} d\chi(y) = \chi(\phi \cap B(z,\epsilon)).$$

The local conical structure of definable sets ([**17**], Thm. 4.10, [**18**]) implies that the right hand side converges to $\phi(z)$ as ϵ tends to 0, i.e. $\phi_\epsilon \to \phi$ pointwise. Using Thom's isotopy lemma ([**37**]) we get that, for all small enough $\epsilon > 0$, ϕ_ϵ is constant near x. From Theorem 3.1 it follows $x \notin \pi_1(\mathrm{spt}(T_\epsilon))$ and

$$\phi_\epsilon(x) = [(\pi_1)_* T_\epsilon].$$

Letting ϵ tend to 0 on both sides finishes the proof. □

Let $\rho_x : V \setminus \{x\} \to S(x,1)$ be the radial projection and $\rho_x^* dv$ be the pull-back of the volume form on $S(x,1)$. Then for any cycle A on V with support in $V \setminus \{x\}$ we have

$$[A] = \frac{1}{s_{n-1}} A(\rho_x^* dv).$$

Here s_{n-1} is the volume of the $n-1$-dimensional sphere.
It follows from the previous theorem that

$$\phi(x) = \frac{1}{s_{n-1}} \lim_{\epsilon \to 0^+} T((\rho_x \circ \pi_1 \circ \exp^\epsilon)^* dv).$$

As our argument above shows, the support of T is for small $\epsilon > 0$ disjoint from the singular set of the differential form $(\rho_x \circ \pi_1 \circ \exp^\epsilon)^* dv$ (which is given by the set $\{(z,v) \in SV : z + \epsilon v = x\}$).
EXAMPLE. Let $X \subset V$ be a compact, definable submanifold. Theorem 3.2 and some elementary topological arguments imply that the normal cycle of T is given by integration over the unit normal bundle of X (which carries a canonical orientation). Another way to see this is to use Morse theory, see [**35**]. Similarly, using stratified Morse theory ([**29**]), one can show that the normal cycle of a definable compact subset of V can be described explicitly in terms of Morse indices associated to height functions, see [**13**].

CHAPTER IV

Tensor-valued measures

Notations. In this chapter, we will use the following conventions and notations.
V will denote an n-dimensional oriented Euclidean vector space, $S(V)$ its unit sphere and $SV = V \times S(V)$ the sphere bundle over V.
For simplicity, we will formulate the definition of the measures $\Lambda_{k,d}$ and their properties only for compact definable sets $X \subset V$. The normal cycle of X will be denoted by $\tilde{X}$. Since only the existence of a normal cycle is used, the measures $\Lambda_{k,d}$ are also well-defined for compact submanifolds with or without boundary and for compact convex sets.

Definition 0.3. *The Kulkarni-Nomizu product of two symmetric bilinear forms h and g on some vector space V is the tensor $h \cdot g \in \operatorname{Sym}^2 \Lambda^2 V^*$ defined by*

$$\begin{aligned} h \cdot g(x, y, v, w) := &\, h(x, v)g(y, w) + h(y, w)g(x, v) - \\ &- h(x, w)g(y, v) - h(y, v)g(x, w) \end{aligned}$$

for all $x, y, v, w \in V$.

The Riemann tensor of a Riemannian manifold (M, g) is denoted by R, the Ricci tensor by ric and the scalar curvature by s. We set $E := \frac{s}{2}g - \operatorname{ric}$ the Einstein tensor and call $\hat{R} := R - \operatorname{ric} \cdot g + \frac{s}{4} g \cdot g$ the *modified Riemann tensor.* If $m = \dim M$, then $\operatorname{tr} E = \frac{m-2}{2}s$, $\operatorname{tr}_{2,4} \hat{R} = (m-3)E$.
The volume element of (M, g) is denoted by μ_g (or just μ).
Let v be a normal vector field defined on a neighborhood of $x \in M$, where $M \subset V$ is a submanifold. We denote by $l_v(X, Y) := \langle D_X v, Y\rangle, X, Y \in T_x M$ the second fundamental form in direction v. It is a symmetric bilinear form which depends only on the value of v at x.
Let $\mathcal{S}_n$ denote the set of permutations of $\{1, \ldots, n\}$ and $\operatorname{sgn}(\pi)$ the sign of a permutation. The volume of the n-dimensional unit sphere is denoted by s_n, the volume of the n-dimensional unit ball by b_n.
To shorten notation, we will write $dx_{i_1 \ldots i_k}$ instead of $dx_{i_1} \wedge \ldots \wedge dx_{i_k}$ and similarly for y. Also, for vectors $e_{i_1}, \ldots, e_{i_k}$ we abbreviate $e_{i_1 \ldots i_k} := e_{i_1} \wedge \ldots \wedge e_{i_k}$.

1. Tensor-valued measures

The flat norm of a compactly supported definable current $T \in \mathcal{D}_{n-1}(SV)$ is defined by

$$\mathcal{F}(T) = \inf\{\mathbf{M}(A) + \mathbf{M}(B) : A + \partial B = T\},$$

where A and B run over all compactly supported definable currents.
Remark. In order to define the flat norm of an arbitrary compactly supported rectifiable current, one lets A and B run over all compactly supported rectifiable currents. Using the Deformation Theorem, one sees that for definable T both definitions agree.
The flat norm induces a distance on the space of compactly supported rectifiable currents on SV, given by $d(T_1, T_2) := \mathcal{F}(T_1 - T_2)$.
By the normal cycle construction of Chapter III, there is an injection from the space of compact definable subsets of V into $\mathcal{R}_{n-1}(SV)$. Via this embedding, we define the flat distance and the flat topology on the space of compact definable subsets of V. By a result of Zähle ([**44**]), this topology restricts to the classical Hausdorff topology on convex bodies.
The aim of this section is to define canonical tensor-valued measures related to X by integrating tensor-valued differential forms over the normal cycle $\tilde{X}$.
After choice of an orthogonal base $e_1, \ldots, e_n$ of V, and using standard coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n)$, we have

$$SV = \left\{ (x_1, \ldots, x_n, y_1, \ldots, y_n) : \sum_{i=1}^n y_i^2 = 1 \right\}.$$

Define for $0 \leq d \leq k \leq n-1$ the following $n-1$-form on SV with values in $\otimes^{2d} V$:

$$\Phi_{k,d} := C_{n,k,d} \sum_{\substack{s_1,\ldots,s_d=1,\ldots,n \\ \pi \in \mathcal{S}_n}} \operatorname{sgn}(\pi) y_{\pi(n)} dx_{s_1 \ldots s_d \pi(d+1) \ldots \pi(k)} \wedge dy_{\pi(k+1) \ldots \pi(n-1)} \otimes$$
$$\otimes e_{s_1} \otimes \ldots \otimes e_{s_d} \otimes e_{\pi(1)} \otimes \ldots \otimes e_{\pi(d)}.$$

The constant is given by

$$C_{n,k,d} := \frac{(-1)^{n+1}}{s_{n-k-1}(k-d)!d!(n-k-1)!}.$$

It can be checked (compare Theorem 1.1, c)) that the form is independent of the choice of the coordinates.
Let X be a compact definable subset of V. Define measures $\Lambda_{k,d}(X, -)$ on V with values in $\otimes^{2d} V$ by setting for $0 \leq d \leq k \leq n-1$

$$\Lambda_{k,d}(X, B) = \tilde{X} \llcorner \pi_1^{-1} B(\Phi_{k,d}) \in \otimes^{2d} V \qquad \forall B \subset V \text{ Borel}$$

and

$$\Lambda_{n,d}(X,B) := \mathrm{vol}_n(X \cap B) \sum_{s_1,\ldots,s_d=1}^{n} (e_{s_1} \wedge \ldots \wedge e_{s_d}) \otimes (e_{s_1} \wedge \ldots \wedge e_{s_d}).$$

By Theorem III.3.1, $\Lambda_{k,d}(X,-)$ is concentrated on X. We also set $\Lambda_{k,d}(X) := \Lambda_{k,d}(X,X)$.
The case $d=0$ is well-known, $\Lambda_{k,0}(X,-)$ is the k-th *Lipschitz-Killing measure* of X ([**26**], [**7**]).

THEOREM 1.1. *Let $X \subset V$ be compact and definable. Let $B \subset V$ be a Borel set. Then*

a) *Valuation property:*

$$\Lambda_{k,d}(X_1,B) + \Lambda_{k,d}(X_2,B) = \Lambda_{k,d}(X_1 \cap X_2,B) + \Lambda_{k,d}(X_1 \cup X_2,B);$$

b) *Translation invariance: $\Lambda_{k,d}(X+t,B+t) = \Lambda_{k,d}(X,B)$ for all $t \in V$;*

c) *Rotation covariance:*

$$\Lambda_{k,d}(\rho X,\rho B) = \rho \Lambda_{k,d}(X,B)$$

for all $\rho \in O(V)$;

d) *Continuity: If $X_i \to X$ in the flat topology, then $\Lambda_{k,d}(X_i,-)$ converges weakly to $\Lambda_{k,d}(X,-)$;*

e) *Homogeneity: Let $\lambda > 0$. Then $\Lambda_{k,d}(\lambda X,\lambda B) = \lambda^k \Lambda_{k,d}(X,B)$;*

f) *Let $\mathrm{tr}_{d,2d} : \otimes^{2d} V \to \otimes^{2d-2} V, d \geq 1$ denote contraction of the d-th and the $2d$-th coordinate. Then*

$$\mathrm{tr}_{d,2d} \Lambda_{k,d}(X,B) = \frac{k-d+1}{d} \Lambda_{k,d-1}(X,B).$$

PROOF. If $k=n$, then all properties are immediate. Let us assume $k<n$.
a) The valuation property follows from $\tilde{X}_1 + \tilde{X}_2 = \widetilde{X_1 \cap X_2} + \widetilde{X_1 \cup X_2}$ (compare [**26**]).
b) The forms $\Phi_{k,d}$ are invariant under the contactomorphism $\phi : SV \to SV, (x,y) \mapsto (x+t,y)$. From Corollary III.2.5 we get $\widetilde{X+t} = \phi_* \tilde{X}$ and thus

$$\Lambda_{k,d}(X+t,B+t) = \widetilde{X+t}(1_{\phi\pi^{-1}B} \wedge \Phi_{k,d}) = \phi_* \tilde{X}(\phi^{-1*} 1_{\pi^{-1}B} \wedge \Phi_{k,d}) = \\ = \tilde{X}(1_{\pi^{-1}B} \wedge \Phi_{k,d}) = \Lambda_{k,d}(X,B).$$

c) ρ induces a contactomorphism $\tilde{\rho} : SV \to SV, (x,y) \mapsto (\rho x, \rho y)$ and, by Proposition III.2.3, $\widetilde{\rho X} = \det(\rho) \tilde{\rho}_* \tilde{X}$.
Letting $\widetilde{dx}_i := \sum_{j=1}^n dx_j \otimes e_j \otimes e_i$, we get, up to a permutation of the factors in the tensorial part,

$$\Phi_{k,d} = C_{n,k,d} \sum_{\pi \in \mathcal{S}_n} \mathrm{sgn}(\pi) y_{\pi(n)} \widetilde{dx}_{\pi(1)\ldots\pi(d)} \wedge dx_{\pi(d+1)\ldots\pi(k)} \wedge dy_{\pi(k+1)\ldots\pi(n-1)}.$$

If ρ is given by the matrix $(\rho_{ij})_{i,j=1}^n$, then $\tilde\rho^*\widetilde{dx_i} = \sum_{j=1}^n \rho_{ij}\rho\circ\widetilde{dx_j}$. Since also $\tilde\rho^* dx_i = \rho_{ij}dx_j, \tilde\rho^* dy_i = \rho_{ij}dy_j$, we get $\tilde\rho^*\Phi_{k,d} = \det(\rho)\rho\circ\Phi_{k,d}$.
It follows that

$$\begin{aligned}\Lambda_{k,d}(\rho X, \rho B) &= \widetilde{\rho X}(1_{\pi^{-1}\rho B}\wedge\Phi_{k,d}) = \det(\rho)\tilde X((\tilde\rho)^* 1_{\pi^{-1}\rho B}\wedge(\tilde\rho)^*\Phi_{k,d}) = \\ &= \tilde X(1_{\pi^{-1}B}\wedge\rho\circ\Phi_{k,d}) = \rho\tilde X(1_{\pi^{-1}B}\wedge\Phi_{k,d}) = \rho\Lambda_{k,d}(X,B).\end{aligned}$$

d) The map $X\mapsto\tilde X$ is, by definition of the flat topology, continuous, and the statement follows.
e) Let $\phi_\lambda : SV\to SV, (x,y)\mapsto(\lambda x, y)$. Then ϕ_λ is a contactomorphism and, again by Proposition III.2.3, $\widetilde{\lambda X} = (\phi_\lambda)_*\tilde X$. The statement now follows from $\phi_\lambda^*\Phi_{k,d} = \lambda^k\Phi_{k,d}$.
f) Obvious. □

2. Symmetry and flatness properties

PROPOSITION 2.1.

$$\Lambda_{k,d}(X,B)\in \mathrm{Sym}^2\Lambda^d V.$$

PROOF. Again, the statement is trivial if $k=n$, so we suppose $k<n$.
Let $\mathcal{I}$ be the ideal generated by α and $d\alpha$. Since $\tilde X\llcorner\alpha = \tilde X\llcorner d\alpha = 0$ for normal cycles, it suffices to show that $\Phi_{k,d}$ has, modulo $\mathcal{I}$, values in $\mathrm{Sym}^2\Lambda^d V$. It is clear by inspecting the form $\Phi_{k,d}$ that its values are in $\Lambda^d V\otimes\Lambda^d V$. By $SO(n)$-covariance, it is enough to show that the values of $\Phi_{k,d}$ at the point $y=(0,\dots,0,1)$ are in $\mathrm{Sym}^2\Lambda^d V$. Note that, at that point, $\alpha = dx_n$ and $\sum_{s=1}^{n-1} dy_s\wedge dx_s = d\alpha - dy_n\wedge\alpha\in\mathcal{I}$.
Define for $i_1,\dots,i_k; j_1,\dots,j_{n-k-1}; k_1,\dots,k_d; l_1,\dots,l_d\in\{1,\dots,n-1\}$

$$A(i_1,\dots,i_k;j_1,\dots,j_{n-k-1};k_1,\dots,k_d;l_1,\dots,l_d) :=$$
$$\sum_{\pi\in\mathcal{S}_{n-1}} \mathrm{sgn}(\pi)dx_{\pi(i_1)\dots\pi(i_k)}\wedge dy_{\pi(j_1)\dots\pi(j_{n-k-1})}\otimes e_{\pi(k_1)\dots\pi(k_d)}\otimes e_{\pi(l_1)\dots\pi(l_d)}.$$

Define for $0\le j\le d$

$$\begin{aligned}A_j := A(1,\dots,j,n-d+j,\dots,n-1,d+1,\dots,k;k+1,\dots,n-1;\\ 1,\dots,j,n-d+j,\dots,n-1;1,\dots,d).\end{aligned}$$

By using symmetry properties and performing several changes of variables in $\mathcal{S}_{n-1}$, we see that $\Phi_{k,d}$ is a linear combination of $A_0,\dots,A_d$. It is therefore enough to show that each A_j is symmetric modulo $\mathcal{I}$. If $n-d+j\le k$, then $A_j=0$. Let us now suppose that $n-d+j>k$.
Define for $0\le l\le d-j$

$$\begin{aligned}A_{j,l} := A(&1,\dots,j,n-d+j,\dots,n-1,d+1,\dots,k;k+1,\dots,n-1;\\ &1,\dots,j+l,n-d+j+l,\dots,n-1;\\ &1,\dots,j,n-d+j,\dots,n-d+j+l-1,j+l+1,\dots,d).\end{aligned}$$

Then A_j is symmetric modulo $\mathcal{I}$ if and only if $A_{j,0} \equiv A_{j,d-j} \mod \mathcal{I}$.
For $0 \leq l \leq d-j-1$, there are terms $dx_{\pi(n-d+j+l)}$ and $dy_{\pi(n-d+j+l)}$ in $A_{j,l}$. If we replace $\pi(n-d+j+l)$ in both terms by $\pi(i)$ and sum over all $i = 1, \ldots, n-1$, we get an element in $\mathcal{I}$. The terms for $i \in \{1, \ldots, j, d+1, \ldots, n-1\} \setminus \{n-d+j+l\}$ vanish trivially.
The terms for $i = j+1, \ldots, j+l$ and $i = n-d+j+l$ are all equal to $A_{j,l}$, whereas the terms for $i = j+l+1, \ldots, d$ are all equal to $-A_{j,l+1}$. Therefore $(l+1)A_{j,l} \equiv (d-j-l)A_{j,l+1} \mod \mathcal{I}$. We deduce that

$$\binom{d-j}{l} A_{j,l} \equiv \binom{d-j}{l+1} A_{j,l+1} \mod \mathcal{I}, \quad l = 0, \ldots, d-j-1,$$

and thus $A_{j,0} \equiv A_{j,d-j} \mod \mathcal{I}$. □

Corollary 2.2. *If $X \subset W$ for some linear subspace $W \subset V$, then for all Borel sets $B \subset V$*

$$\Lambda_{k,d}(X, B) \in \operatorname{Sym}^2 \Lambda^d W.$$

Proof. By Theorem 1.1 c), it suffices to prove this for $W := \{x_n = 0\}$, since a k-dimensional space $k < n$ is the intersection of all hyperplanes containing it.
If $X \subset W$, then $\operatorname{spt} \tilde{X} \subset W \times V$ by Theorem III.3.1. In particular $\tilde{X} \llcorner dx_n = 0$. We deduce by inspecting the form $\Phi_{k,d}$ that all terms where one of the $s_i, i = 1, \ldots, d$ equals n vanish on $\tilde{X}$. This implies $\Lambda_{k,d}(X, B) \in \Lambda^d W \otimes \Lambda^d V$. Since $\Lambda^d W \otimes \Lambda^d V \cap \operatorname{Sym}^2 \Lambda^d V = \operatorname{Sym}^2 \Lambda^d W$, the result follows from Proposition 2.1. □

The next theorem generalizes this corollary.

Theorem 2.3. *There exists a C^2-cell decomposition of X such that, for each cell D, $\Lambda_{k,d}(X, -)|_D$ is a $\operatorname{Sym}^2 \Lambda^d TD$-valued measure.*

Proof. Fix C^2-cell decompositions of $\operatorname{spt} \tilde{X}$ and X, compatible with π_1.
If $k = n$, then $\Lambda_{n,d}(X, -)|_D$ vanishes for all cells of dimension less than n, so there is nothing to prove.
Suppose $k < n$. Pairing $\tilde{X}$ with $\Phi_{k,d}$ yields a $\operatorname{Sym}^2 \Lambda^d V$-valued measure on SV. Its push-forward to V under π_1 is a priori a $\operatorname{Sym}^2 \Lambda^d V$-valued measure. Since for almost all $(x, y) \in \operatorname{spt} \tilde{X}, x \in D$ the tangent plane $W := T_{(x,y)} \operatorname{spt} T$ annihilates α and $d\alpha$ and, moreover, its projection under π_1 equals $T_x D$, the proof is finished by the following lemma.

Lemma 2.4. *Let $(x, y) \in SV$ and let W be an oriented $n-1$-dimensional plane in $T_{(x,y)} SV$ such that $W \llcorner \alpha = W \llcorner d\alpha = 0$. Then*

$$\Phi_{k,d}(W) \subset \operatorname{Sym}^2 \Lambda^d \pi_1(W).$$

Proof. Since $\Phi_{k,d}$ is $SO(n)$-covariant, we can assume that $\pi_1(W) = \{x \in \mathbb{R}^n : x_{l+1} = \ldots = x_n = 0\}$ $(l = \dim \pi_1(W))$. But then $W \llcorner dx_i = 0$

for $i = l+1, \ldots, n$, and it follows by inspecting the forms $\Phi_{k,d}$ that $\Phi_{k,d}(W) \subset \Lambda^d \pi_1(W) \otimes \Lambda^d V$.
As was shown in the proof of Proposition 2.1, $\Phi_{k,d}(W) \subset \operatorname{Sym}^2 \Lambda^d V$.
The statement of the lemma now follows from

$$\Lambda^d \pi_1(W) \otimes \Lambda^d V \cap \operatorname{Sym}^2 \Lambda^d V = \operatorname{Sym}^2 \Lambda^d \pi_1(W).$$

□

□

3. Comparison with Alesker invariants

Define the $n-1$-form $\Phi'_{k,1}$ on SV with values in $\operatorname{Sym}^2 V$ by

$$\Phi'_{k,1} = \frac{(-1)^{n+1}}{s_{n-k-1}(k-1)!(n-k-1)!} \times \\ \times \sum_{s=1}^{n} \sum_{\pi \in \mathcal{S}_n} \operatorname{sgn}(\pi) y_{\pi(n)} dx_{s\pi(2)\ldots\pi(k)} \wedge dy_{\pi(k+1)\ldots\pi(n-1)} e_s e_{\pi(1)}.$$

By Theorem 2.1, $\Phi_{k,1} \equiv \Phi'_{k,1} \mod \mathcal{I}$.
Define $n-1$-forms on SV with values in $\operatorname{Sym}^2 V$ by

$$\Psi_1 := \Phi_{k,0} \otimes (e_1^2 + \cdots + e_n^2) \quad \text{for } k = 1, \ldots, n-1;$$

$$\Psi_2 := \Phi_{k,0} \otimes (y_1 e_1 + \cdots + y_n e_n)^2 \quad \text{for } k = 1, \ldots, n-1;$$

$$\Psi_3 := \frac{(-1)^{n+1}}{s_{n-k-1} k!(n-k-1)!} \times \\ \times \sum_{s=1}^{n} \sum_{\pi \in \mathcal{S}_n} \operatorname{sgn}(\pi) y_{\pi(n)} y_s dx_{\pi(1)\ldots\pi(k)} \wedge dy_{\pi(k+1)\ldots\pi(n-2)} e_s e_{\pi(n-1)} \\ \text{for } k = 1, \ldots, n-2, \quad \Psi_3 := 0 \text{ for } k = n-1;$$

$$\Psi_4 := \frac{(-1)^{n+1}}{s_{n-k-1} k!(n-k-1)!} \times \\ \times \sum_{s=1}^{n} \sum_{\pi \in \mathcal{S}_n} \operatorname{sgn}(\pi) y_{\pi(n)} y_s dx_{\pi(2)\ldots\pi(k)} \wedge dy_{\pi(k+1)\ldots\pi(n-1)} e_{\pi(1)} e_s \\ \text{for } k = 1, \ldots, n-1.$$

Lemma 3.1.

$$\Phi'_{k,1} + (n-k-1)(-1)^n d\Psi_3 = \Psi_1 - (n-k)\Psi_2 + (n-k)\alpha \wedge \Psi_4.$$

Proof. All the forms involved are $SO(n)$-equivariant $\operatorname{Sym}^2 V$-valued $n-1$-forms. It thus suffices to verify the equation at $y = (0, \ldots, 0, 1)$, which is a lengthy, but simple counting of terms. □

Theorem 3.2. *If $X \subset V$ is compact and definable and $1 \leq k \leq n-1$, then*

$$\Lambda_{k,1}(X) = \tilde{X}(\Psi_1) - (n-k)\tilde{X}(\Psi_2).$$

PROOF. Since $\Lambda_{k,1}(X) = \tilde{X}(\Phi_{k,1}) = \tilde{X}(\Phi'_{k,1})$, the equation follows from the above lemma and $\partial\tilde{X} = \tilde{X} \llcorner \alpha = 0$. $\square$

The theorem (at least for $X = K$ convex) also follows by a recent result of Alesker ([**2**]) which is based on [**1**]. He studied the space of all translation invariant, rotation covariant continuous valuations (on compact convex sets K) with values in $\operatorname{Sym}^2 V$. His theorem implies that the subspace consisting of valuations which are homogeneous of degree k is generated by $K \mapsto \tilde{K}(\Psi_1)$ and $K \mapsto \tilde{K}(\Psi_2)$. Since $\Lambda_{k,1}$ is such a valuation, it is a linear combination of these two basic valuations and the constants can be easily computed by plugging in examples.
We remark that this is a *global* result. The corresponding measures are different, as can be seen on easy examples. The reason is that $\tilde{K}(d\Psi_3) = 0$ only globally, but in general $\tilde{K} \llcorner B(d\Psi_3) \neq 0$ for a Borel set B. Since we are interested in curvature *measures* and not only in global curvatures, we have to use the differential form $\Phi_{k,1}$ instead of the linear combination of Ψ_1 and Ψ_2.

4. Submanifolds

Let $M \subset V$ be a compact submanifold with scalar curvature s, Einstein tensor E, modified Riemann tensor $\hat{R}$ and volume form μ (all with respect to the induced Riemannian metric on M). Using the Euclidean structure, we can define the dual $(2,0)$-tensor $E^\#$ and the dual $(4,0)$-tensor $\hat{R}^\#$ on M.

THEOREM 4.1. *Let $M \subset V$ be a compact m-dimensional submanifold and $B \subset V$ a Borel subset. Then*

a) $\Lambda_{m-2,0}(M,B) = \frac{1}{4\pi}\int_{M\cap B} s\mu$ *if* $m \geq 2$*;*
b) $\Lambda_{m-2,1}(M,B) = \frac{1}{2\pi}\int_{M\cap B} E^\#\mu$ *if* $m \geq 3$*;*
c) $\Lambda_{m-2,2}(M,B) = \frac{1}{4\pi}\int_{M\cap B} \hat{R}^\#\mu$ *if* $m \geq 4$.

REMARK.

- We could also continue, but there is not much additional information in the following terms. For instance, $\Lambda_{m-2,3}$ is related to the $(0,6)$-tensor $R\cdot g - \frac{1}{2}\operatorname{ric}\cdot g\cdot g + \frac{s}{12} g\cdot g\cdot g$. Here the dot means the following: the wedge product on $\Lambda^* T^*M$ induces a commutative product on $\sum_k \left(\Lambda^k T^*M \otimes \Lambda^k T^*M\right)$ which restricts to a product, called *Kulkarni-Nomizu product* on $\oplus_k \operatorname{Sym}^2 \Lambda^k T^*M$ (see Definition 0.3 for the case $k=1$). On the other hand, $\Lambda_{i,j}$ with $i < m-2$ yields to higher order curvature terms. They certainly contain a lot of information, but it is hard to extract it.
- The trace of the Einstein tensor is $\frac{m-2}{2}s$, the trace of $\hat{R}$ is $\operatorname{tr}_{2,4}\hat{R} = (m-3)E$. Furthermore, both tensor fields are divergence free: $\delta E = \delta\hat{R} = 0$. These equations are easily obtained

by taking traces of the differential Bianchi identity. Furthermore, E vanishes if $n = 2$ and $\hat{R}$ vanishes if $m \leq 3$. It seems that, in contrast to the Einstein tensor, the tensor $\hat{R}$ has no clear geometrical meaning and that it was not yet studied in Riemannian geometry.

- Hilbert's variational formula relates scalar curvature and Einstein tensor of a Riemannian manifold. A generalization of this formula, relating $\Lambda_{k,0}$ and $\Lambda_{k,1}$ for $1 \leq k \leq n$ is proved in [**5**].

Proof. Only the proof of b) will be given. The proof of c) is very similar, but a bit longer. a) is well-known, compare [**3**].
Since the statement is a local one, we can suppose that M is oriented and that there is a base of normal vector fields $\nu_{m+1}, \ldots, \nu_n$ along M. By changing ν_n to $-\nu_n$ if necessary, we can suppose that, if $e_1 \wedge \ldots \wedge e_m \in \Lambda^m T_x M$ is positive, then $e_1 \wedge \ldots \wedge e_m \wedge \nu_{m+1}(x) \wedge \ldots \wedge \nu_n(x)$ is positive in $\Lambda^n T_x V$.
Let S^{n-m-1} be the unit sphere in $\mathbb{R}^{n-m}$ with coordinates $t_{m+1}, \ldots, t_n$, $t_{m+1}^2 + \cdots + t_n^2 = 1$. Define $H : M \times S^{n-m-1} \to V \times S(V), (x,t) \mapsto (x, t_{m+1}\nu_{m+1}(x) + \ldots + t_n \nu_n(x))$. The normal cycle of M is then given by

$$\tilde{M} = (-1)^m H_* \left([[M]] \times [[S^{n-m-1}]]\right)$$

(compare Section III.3, one can check the sign by considering m-dimensional spheres).
Let us compute $H^* \Phi'_{m-2,1}$ in a point $(\bar{x}, \bar{t}) \in M \times S^{n-m-1}$. Since the definition of $\Phi'_{m-2,1}$ is independent of the chosen positively oriented coordinate system, we can choose one for which the computation is particularly easy.
We choose a positive orthonormal basis $(e_1, \ldots, e_n)$ of V in such a way that $e_i \in T_{\bar{x}} M, i = 1, \ldots, m$ are the principal curvature directions with respect to the normal vector $\nu := \bar{t}_{m+1}\nu_{m+1}(\bar{x}) + \ldots + \bar{t}_m \nu_m(\bar{x})$. We suppose that $e_1 \wedge \ldots \wedge e_m$ is positive. Let $\lambda_i = \lambda_i(\nu), i = 1, \ldots, m$ denote the corresponding principal curvatures and by $(x_1, \ldots, x_n)$ the coordinates with respect to this basis.
Let $\nu_k(x) = (\nu_k^1(x), \ldots, \nu_k^n(x))$. Since $\nu_k(\bar{x}) \perp T_{\bar{x}} M$, we have $\nu_k^i(\bar{x}) = 0$ for $i = 1, \ldots, m$.
For $i, j = 1, \ldots, m$, the choice of the basis implies that

$$\sum_{k=1}^{m} \bar{t}_k \frac{\partial \nu_k^j(\bar{x})}{\partial x_i} = \langle D_{e_i} \nu, e_j \rangle = \lambda_i \delta_{ij}.$$

For all $j = 1, \ldots, n$, we have $H^* x_j = x_j, H^* y_j = \sum_{k=m+1}^n t_k \nu_k^j$ from which we deduce (using $dx_j = 0$ on $T_{\bar{x}} M$ for $j = m+1, \ldots, n$ and the above equation) that

$$H^* dx_j|_{(\bar{x}, \bar{t})} = \begin{cases} dx_j|_{\bar{x}} & j = 1, \ldots, m \\ 0 & j = m+1, \ldots, n \end{cases}$$

and

$$H^*dy_j|_{(\bar{x},\bar{t})} = \begin{cases} \lambda_j dx_j|_{\bar{x}} \text{ if } j \leq m \\ \sum_{k=m+1}^{n} \left(\nu_k^j(\bar{x}) dt_k|_{\bar{t}} + \bar{t}_k \sum_{i=1}^{m} \frac{\partial \nu_k^j(\bar{x})}{\partial x_i} dx_i \right) \text{ else.} \end{cases}$$

Recall that, with $C_1 := \frac{(-1)^{n+1}}{s_{n-m+1}(m-3)!(n-m+1)!}$,

$$\begin{aligned} \Phi'_{n-2,1} &= C_1 \sum_{s=1}^{n} \sum_{\pi \in \mathcal{S}_n} \operatorname{sgn}(\pi) y_{\pi(n)} dx_{s\pi(2)\ldots\pi(m-2)} \wedge dy_{\pi(m-1)\ldots\pi(n-1)} e_s e_{\pi(1)} \\ &= C_1 \sum_{\pi \in \mathcal{S}_n} \operatorname{sgn}(\pi) y_{\pi(n)} dx_{\pi(1)\ldots\pi(m-2)} \wedge dy_{\pi(m-1)\ldots\pi(n-1)} e^2_{\pi(1)} + \\ &\quad + \text{ terms containing } dx_j \wedge dy_j \text{ or } y_j dx_j \text{ for some } j = 1, \ldots, n. \end{aligned}$$

Since $H^*y_j|_{(\bar{x},\bar{t})} = 0$ for $j = 1, \ldots, m$ and $H^*dx_j|_{(\bar{x},\bar{t})} = 0$ for $j = m+1, \ldots, n$, we can omit terms containing $y_j dx_j$ for some j.
Similarly, if $j \leq m$ then $H^*dy_j|_{(\bar{x},\bar{t})} = \lambda_j dx_j|_{\bar{x}}$; and if $j > m$ then $H^*dx_j|_{(\bar{x},\bar{t})} = 0$. In both cases $H^*(dx_j \wedge dy_j)|_{(\bar{x},\bar{t})} = 0$, which means that we only have to take into account the first term.
Let A be the component of degree $(m, n-m-1)$ of $H^*\Phi'_{m-2,1}|_{(\bar{x},\bar{t})}$. Then, with $C_2 := \binom{n-m+1}{2} C_1$,

$$\begin{aligned} A = C_2 \sum_{\pi \in \mathcal{S}_m} \operatorname{sgn}(\pi)\ \lambda_{\pi(m-1)} \lambda_{\pi(m)} dx_{\pi(1)\ldots\pi(m)}\big|_{\bar{x}} \wedge \\ \wedge \underbrace{\sum_{\pi' \in \mathcal{S}'_{n-m}} \operatorname{sgn}(\pi') \sum_{k_{m+1},\ldots,k_n = m+1}^{n} \nu_{k_{m+1}}^{\pi'(m+1)} \cdots \nu_{k_n}^{\pi'(n)}\ t_{k_n} dt_{k_{m+1}\ldots k_{n-1}}\big|_{\bar{t}}}_{=:A_1} \otimes e^2_{\pi(1)}. \end{aligned} \tag{5}$$

Now $e_1, \ldots, e_m, \nu_{m+1}(x), \ldots, \mu_n(x)$ is a positive orthonormal basis and we can compute A_1 as follows.

$$\begin{aligned} A_1 &= \sum_{\pi' \in \mathcal{S}'_{n-m}} \operatorname{sgn}(\pi') \sum_{k_{m+1},\ldots,k_n = m+1}^{n} \nu_{k_{m+1}}^{\pi'(m+1)} \cdots \nu_{k_n}^{\pi'(n)} t_{k_n} dt_{k_{m+1}\ldots k_{n-1}}\Big|_{\bar{t}} \\ &= \sum_{\pi', \tau \in \mathcal{S}'_{n-m}} \operatorname{sgn}(\pi') \nu_{\tau(m+1)}^{\pi'(m+1)} \cdots \nu_{\tau(n)}^{\pi'(n)} t_{\tau(n)} dt_{\tau(m+1)\ldots\tau(n-1)}\Big|_{\bar{t}} \\ &= \sum_{\tau \in \mathcal{S}'_{n-m}} \operatorname{sgn}(\tau)\ t_{\tau(n)} dt_{\tau(m+1)\ldots\tau(n-1)}\big|_{\bar{t}} \\ &= (n-m-1)!(-1)^{n-m-1} dt\big|_{\bar{t}} \end{aligned}$$

Plugging this into Equation (5) and setting $C_3 := C_2(n-m-1)!(-1)^{n-m-1}$ we obtain

$$\begin{aligned} A &= C_3 \sum_{\pi\in\mathcal{S}_m} \operatorname{sgn}(\pi)\lambda_{\pi(m-1)}\lambda_{\pi(m)} dx_{\pi(1)\dots\pi(m)}\big|_{\bar{x}} \wedge dt \otimes e^2_{\pi(1)} \\ &= C_3\, dx_{1\dots m}|_{\bar{x}} \wedge dt \underbrace{\sum_{\pi\in\mathcal{S}_m} \lambda_{\pi(m-1)}\lambda_{\pi(m)} \otimes e^2_{\pi(1)}}_{=:A_2} . \end{aligned}$$

The term A_2 is given by

$$\begin{aligned} A_2 &= \frac{1}{m-2} \sum_{\pi\in\mathcal{S}_m} \lambda_{\pi(m-1)}\lambda_{\pi(m)} \otimes \left(\sum_{s=1}^m e_s^2 - 2e^2_{\pi(m)} \right) \\ &= 2(m-3)! \sum_{i\neq j=1}^m \lambda_i\lambda_j \left(\sum_{s=1}^m e_s^2 - 2e_i^2 \right) \\ &= 2(m-3)! \Bigg[\sum_{i,j,s=1}^m \left(l_\nu(e_i,e_i) l_\nu(e_j,e_j) - l_\nu(e_i,e_j) l_\nu(e_i,e_j) \right) e_s^2 - \\ &\quad - 2 \left(l_\nu(e_i,e_s) l_\nu(e_j,e_j) - l_\nu(e_i,e_j) l_\nu(e_j,e_s) \right) e_i e_s \Bigg]. \end{aligned}$$

This is an expression for the integrand which is independent of the choice of $e_{m+1},\dots,e_n$ and we can integrate it with respect to t.
Let $l_r := l_{\nu_r}, r = m+1,\dots,n$ be the second fundamental form in direction ν_r. From Gauss equation we obtain

$$\begin{aligned} &\int_{S^{n-m-1}} \left(l_\nu(e_i,e_i) l_\nu(e_j,e_j) - l_\nu(e_i,e_j) l_\nu(e_i,e_j) \right) dt \\ &= \int_{S^{n-m-1}} \sum_{r,r'=m+1,\dots,n} t_r t_{r'} \left(l_r(e_i,e_i) l_{r'}(e_j,e_j) - l_r(e_i,e_j) l_{r'}(e_i,e_j) \right) dt \\ &\quad = \underbrace{\sum_{r=m+1,\dots,n} \left(l_r(e_i,e_i) l_r(e_j,e_j) - l_r(e_i,e_j) l_r(e_i,e_j) \right)}_{=R(e_i,e_j,e_i,e_j)} \underbrace{\int_{S^{n-m-1}} t_r^2 dt}_{=\frac{s_{n-m-1}}{n-m}} . \end{aligned}$$

With $C_4 := 2(m-3)!\frac{s_{n-m-1}}{n-m} C_3$ we thus get

$$\begin{aligned} &\int_{S^{n-m-1}} H^*\Phi'_{m-2,1}|_{(\bar{x},t)} dt \\ &= C_4 \sum_{i,j,s=1}^m \left(R_{\bar{x}}(e_i,e_j,e_i,e_j) e_s^2 - 2R_{\bar{x}}(e_i,e_j,e_s,e_j) e_i e_s \right) \mu_{\bar{x}} = C_4 E^{\#}_{\bar{x}} \mu|_{\bar{x}}. \end{aligned}$$

Using $s_{n-m+1} = \frac{2\pi}{n-m} s_{n-m-1}$ we easily obtain $C_4 = \frac{(-1)^m}{2\pi}$ and b) follows by integration over B. $\square$

Remark. To check that there is no computational error for the constants, we can also look at traces. We have already noticed that $\operatorname{tr} E = \frac{m-2}{2}s$. Theorem 1.1, f) implies that the constants $\frac{1}{2\pi}$ in b) is the right one. In the same way, one can check that $\frac{1}{4\pi}$ is the right constant in c).

Example. Let $M := S^m \subset \mathbb{R}^{m+1}$. Since the sectional curvature of M is 1, we get $s = m(m-1)$, $\operatorname{ric} = (m-1)g, R = \frac{1}{2}g \cdot g$. It follows $E = \binom{m-1}{2}g$ and $\hat{R} = \frac{(m-2)(m-3)}{4}g \cdot g$. At $v \in S^m$, we therefore have $E^\# = \binom{m-1}{2}g^\# = \binom{m-1}{2}(\sum_{s=1}^{m+1} e_s^2 - v^2)$. Theorem 4.1 implies that, for $B \subset S^m$ Borel,

$$\Lambda_{m-2,1}(S^m, B) = \frac{1}{2\pi}\binom{m-1}{2}\int_{S^m \cap B}\left(\sum_{s=1}^{m+1} e_s^2 - v^2\right) dv \in \operatorname{Sym}^2 \mathbb{R}^{m+1}.$$

In particular,

$$\Lambda_{m-2,1}(S^m) = \frac{m(m-1)(m-2)s_m}{4\pi(m+1)}\sum_{s=1}^{m+1} e_s^2.$$

Similarly,

$$\Lambda_{m-2,2}(S^m, B) = C_m \int_{S^m \cap B}\left(\sum_{i,j=1}^{m+1}(e_i \wedge e_j)^2 - 2\sum_{j=1}^{m+1}(v \wedge e_j)^2\right) dv \in \operatorname{Sym}^2 \Lambda^2\mathbb{R}^{m+1}$$

with $C_m = \frac{(m-2)(m-3)}{16\pi}$.

5. Ricci curvature

Theorem 4.1 shows that the measure $\Lambda_{m-2,1}(X, -)$ is a weak generalization of the Einstein tensor to the class of compact definable sets. It is natural to ask if there is a weak notion of Ricci tensor of compact definable sets. The answer is "no", as the next proposition shows.

Proposition 5.1. *There is no map Λ_{ric} associating to compact definable sets symmetric bivectors such that*

a) $\Lambda_{ric}(X) = \frac{1}{2\pi}\int_X \operatorname{ric}^\# \mu_g$ *for all compact m-dimensional submanifolds $X \subset V$ $(m \geq 3)$,*
b) Λ_{ric} *is continuous with respect to the flat topology,*
c) *If $X \subset W$ for some linear subspace, then $\Lambda_{ric}(X) \subset \operatorname{Sym}^2 W$.*

Proof. Suppose Λ_{ric} exists. Then $\Lambda' := \Lambda_{m-2,1} + \Lambda_{ric}$ satisfies b) and c) and $\Lambda'(X) = \frac{1}{4\pi}\int_X sg^\# \mu_g$ for compact m-dimensional submanifolds X.

The rescalings λX, tend for $\lambda \to 0$ in the flat topology to the 1-point space $\{0\}$, counted with multiplicity $\chi(X)$. This follows easily from the homotopy formula for currents.

Let W_1 be an n_1-dimensional Euclidean vector space containing a compact two-dimensional submanifold X of Euler characteristic equal to 1 (e.g. $X = \mathbb{R}P^2$). Then λX converges to the one point set $\{0\}$.
Let W_2 be an n_2-dimensional Euclidean vector space and $Y \subset W_2$ a compact $m-2$-dimensional submanifold. We identify W_2 with $\{0\} \times W_2 \subset W_1 \oplus W_2$. Then $\lambda X \times Y \to Y$ as $\lambda \to 0$ and therefore, by b) and c), $\lim_{\lambda \to 0} \Lambda'((\lambda X) \times Y) = \Lambda'(Y) \in W_2 \otimes W_2$.
From $s_{\lambda X \times Y}(\lambda x, y) = s_{\lambda X}(\lambda x) + s_Y(y)$ $(x \in X, y \in Y)$ and $g_{\lambda X \times Y} = g_{\lambda X} \oplus g_Y$ we deduce that

$$\begin{aligned}
4\pi\Lambda'(\lambda X \times Y) &= \int_{\lambda X \times Y} s_{\lambda X \times Y} g^{\#}_{\lambda X \times Y} \mu_{\lambda X \times Y} \\
&= \mathrm{vol}(Y) \underbrace{\int_{\lambda X} s_{\lambda X} g^{\#}_{\lambda X} \mu_{\lambda X}}_{\in W_1 \otimes W_1} + \underbrace{\int_{\lambda X} s_{\lambda X} \mu_{\lambda X}}_{\in \mathbb{R}} \underbrace{\int_Y g^{\#}_Y \mu_Y}_{\in W_2 \otimes W_2} + \\
&\quad + \underbrace{\int_{\lambda X} g^{\#}_{\lambda X} \mu_{\lambda X}}_{C\Lambda_{2,1}(\lambda X) \to 0} \underbrace{\int_Y s_Y \mu_Y}_{\in \mathbb{R}} + \mathrm{vol}(\lambda X) \underbrace{\int_Y s_Y g^{\#}_Y \mu_Y}_{\in W_2 \otimes W_2}.
\end{aligned}$$

The third summand tends to 0 as $\lambda \to 0$, since $\Lambda_{2,1}(\lambda X) = \lambda^2 \Lambda_{2,1}(X)$ by homogeneity. By Gauss-Bonnet, the trace of the first summand is non-zero and independent of λ, namely $8\pi \,\mathrm{vol}(Y)\chi(X) = 8\pi \,\mathrm{vol}(Y)$. The whole sum tends therefore to some element of $(W_1 \oplus W_2) \otimes (W_1 \oplus W_2)$ with a non-vanishing component in $W_1 \otimes W_1$. Contradiction. □

6. Definable sets

DEFINITION 6.1. a) *A stratification of a set $X \subset V$ is a decomposition of X into a disjoint, locally finite union of submanifolds of V, called strata, such that the boundary of each stratum is a union of strata.*

b) *A stratified subset $X \subset V$ satisfies Whitney's condition B at $x \in X$, if for all pairs S_1, S_2 of strata with $x \in S_1$, the following condition is fulfilled:*

Let $(x_k)_{k \in \mathbb{N}}, (y_k)_{k \in \mathbb{N}}$ be two sequences of points with $x_k \in S_1, y_k \in S_2, x_k \neq y_k, \lim_{k \to \infty} x_k = \lim_{k \to \infty} y_k = x$ such that the lines $\overline{x_k y_k}$ converge to a line L and such that the tangent spaces $T_{y_k} S_2$ converge to a limit space T. Then $L \subset T$.

c) *The space X is said to satisfy condition B if this is the case for each $x \in X$.*

THEOREM 6.2. **Existence of Whitney stratification**
Any definable set $X \subset V$ admits a Whitney-B-stratification.

PROOF. See [**18**]. □

THEOREM 6.3. **Pawłucki's theorem**
Let $X \subset V$ be an m-dimensional Whitney-B-stratified subset. Let $x \in$

X be contained in an $m-1$-dimensional stratum Y. Then there exists an open neighborhood U of x such that $(X \cap U) \setminus Y$ is a C^1-manifold and the germ at x of each of its connected components is C^1-diffeomorphic to the germ at 0 of the set $\{x \in \mathbb{R}^n : x_m > 0, x_{m+1} = \cdots = x_m = 0\}$.

PROOF. See [**36**]. □

It follows that, if x is contained in an $m-1$-dimensional stratum Y, there exists a finite number of outward normal vectors corresponding to the connected components of the germ of $X \setminus Y$ at x. We let w be their sum and call w the *total outward normal vector* . We denote by $l = l_w(x)$ the second fundamental form of Y at x in direction w.
The restriction of g to a stratum Y will be denoted by g_Y. Let $\mathcal{H}_Y^{\dim Y}$ denote its $\dim Y$-dimensional Hausdorff measure. The trace of a bilinear form on a stratum Y (with respect to g_Y) is denoted by tr.
The limit $\theta(X,x) := \lim_{r \to 0^+} \frac{\mathrm{vol}_m(X \cap B(x,r))}{b_m r^m}$ is called *density* of X at x. We define $\eta(x) := \frac{1}{2} + (-1)^m \frac{\chi_{loc}(X,x)}{2} - \theta(X,x)$, where $\chi_{loc}(X,x)$ denotes the local Euler characteristic of X at x. If x belongs to an m-dimensional stratum, then $\chi_{loc}(X,x) = (-1)^m$ and $\theta(X,x) = 1$, hence $\eta(X,x) = 0$. If x belongs to a stratum of dimension $m-1$ and if the germ of $X \setminus Y$ at x has k connected components, then, by Pawłucki's theorem, $\chi_{loc}(X,x) = (-1)^{m-1} + k(-1)^m$ and $\theta(X,x) = \frac{k}{2}$. Again $\eta(X,x) = 0$. But on lower-dimensional strata, $\eta \neq 0$ in general.
Let T_i be a map which associates to each i-dimensional stratum Y a tensor field on Y. For instance, s, l and η are such maps, from which we can build other ones by taking traces and by multiplying with the metric tensor.
We say that a tensor-valued measure $\Lambda(X,-)$ is represented by the $m+1$-tuple $(T_0, T_1, \ldots, T_m)$ if

$$\Lambda(X,B) = \sum_{i=0}^{m} \sum_{Y, \dim Y = i} \int_{Y \cap B} T_i(Y) d\mathcal{H}_Y^{\dim Y} \text{ for all Borel subsets } B \subset V.$$

THEOREM 6.4. *Let $X \subset V$ be compact definable of dimension m with a fixed Whitney-B-stratification. Then*

a) *for $m \geq 2$, $\Lambda_{m-2,0}(X,-)$ is represented by*
$$\left(0, \ldots, 0, \eta, \frac{1}{2\pi} \operatorname{tr} l, \frac{1}{4\pi} s\right);$$
b) *for $m \geq 3$, $\Lambda_{m-2,1}(X,-)$ is represented by*
$$\left(0, \ldots, 0, \eta \cdot g^{\#}, \frac{1}{2\pi}(\operatorname{tr} l \cdot g - l)^{\#}, \frac{1}{2\pi} E^{\#}\right);$$
c) *for $m \geq 4$, $\Lambda_{m-2,2}(X,-)$ is represented by*
$$\left(0, \ldots, 0, \frac{1}{4}\eta(g \cdot g)^{\#}, \frac{1}{8\pi}(\operatorname{tr} l \cdot g \cdot g - 2l \cdot g)^{\#}, \frac{1}{4\pi} \hat{R}^{\#}\right).$$

PROOF. We refer to [**9**] for the proof. It is similar to the one of Theorem 4.1. □

7. Polyhedral submanifolds

DEFINITION 7.1.
- *The Einstein measure of a closed smooth submanifold $X \subset V$ is the* $\mathrm{Sym}^2 V$*-valued measure given by*
$$E(X,B) := \int_{X\cap B} E^{\#} d\mu_g$$
for all Borel sets $B \subset V$.
- *The Einstein measure of a closed polyhedral submanifold $X \subset V$ of dimension $3 \le m < n$ is the* $\mathrm{Sym}^2 V$*-valued measure defined by*
$$E(X,B) := 2\pi \sum_{Y, \dim Y = m-2} \int_{Y\cap B} (1-\theta(X,x)) g_Y^{\#} d\mathcal{H}_Y^{m-2}(x)$$
for all Borel sets $B \subset V$. Here Y runs over all $m-2$-dimensional faces of X and $\theta(X,x)$ is the density of X at x.

Note that $\chi_{loc}(X,x) = (-1)^m$, since X is a topological manifold. Therefore $\eta_Y(x) = (1-\theta(X,x)), x \in Y$. By Theorem 4.1 b) and Theorem 6.4 b), the Einstein measure of a smooth or polyhedral submanifold X is the same as $2\pi\Lambda_{m-2,1}(X,-)$, but the above expressions are more explicit.

The fatness of a k-simplex $Y \subset V$ with vertices $v_0,\dots,v_k$ is defined as
$$\frac{\mathcal{H}^k(Y)}{\max_{i\neq j}\|v_i - v_j\|^k}.$$
The fatness of a triangulated piecewise linear space in V is the minimum of the fatnesses of the simplexes of the triangulation. Finally, the fatness of a piecewise linear space is the supremum over the fatnesses of all its triangulations (compare [**24**]).

Let M be a compact smooth submanifold of V. Then M has positive reach, i.e. there exists $r > 0$ such that for all $x \in V$ with $d(x,M) < r$ there exists a unique $\xi(x) \in M$ with $|x - \xi(x)| = d(x,M)$.

COROLLARY 7.2. *Let $M \subset V$ be a compact smooth submanifold of dimension $3 \le m < n$ and $X_1, X_2, \dots$ a sequence of m-dimensional polyhedral submanifolds such that*

a) *X_i converges for $i \to \infty$ to M in the Hausdorff topology;*
b) *the fatness of X_i remains bounded from below by some constant $c > 0$;*
c) *X_i is closely inscribed in M, i.e. all vertices of X_i are on M, X_i is contained in the domain of ξ and $\xi|_{X_i}$ is one-to-one.*

Then the Einstein measure of X_i tends weakly to the Einstein measure of M.

Proof. By the Main Theorem of [**24**], the normal cycles $\tilde{X}_i$ converge in the flat topology to $\tilde{M}$. Now apply Theorem 1.1 d). $\square$

The analogous statement for the modified Riemann tensor can also be shown with the same proof.
A similar theorem, but concerning intrinsic approximations (i.e. the lengths of the edges are induced by geodesic distances on M), is known for Lipschitz-Killing measures ([**15**]). I do not know how to generalize this to the Einstein measure.
In the case $n = 3$, the Einstein measure was also considered in [**16**], where applications of the piecewise linear approximation of the Einstein measure in computational geometry are presented.

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