UNIVERSIDAD TECNICA
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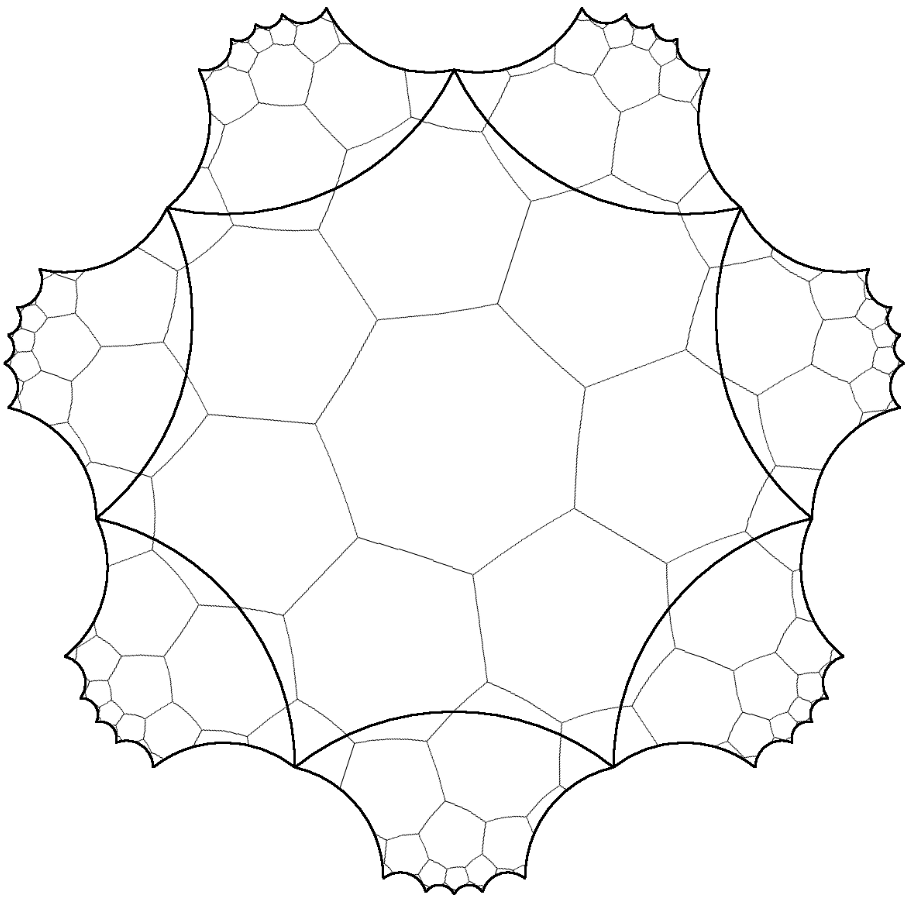
Álgebras de cuaterniones y Grupos fuchsianos aritméticos



Valparaíso, julio de 2012

Resumen

Una superficie de Riemann es una variedad de dimensión 2 en la que los cambios de cartas son funciones holomorfas entre abiertos del plano complejo. Las superficies de Riemann son siempre orientables, y por lo tanto las compactas están caracterizadas topológicamente por su género. Las superficies de Riemann compactas se pueden ver también como curvas algebraicas lisas sobre los complejos, y por lo tanto se puede definir una acción del grupo de Galois $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ sobre el conjunto de las superficies de Riemann compactas mediante la acción de los elementos de Galois en los polinomios con coeficientes complejos.

Otra forma de estudiar superficies de Riemann es desde el punto de vista de la uniformización. Por la teoría de espacios recubridores toda superficie de Riemann es el cociente de una superficie simplemente conexa, llamada recubridor universal, por la acción libre de un subgrupo del grupo de automorfismos de este recubridor. El Teorema de Uniformización nos asegura que toda superficie de Riemann simplemente conexa es isomorfa al plano, a la esfera o al disco unitario, y por lo tanto estos son los únicos posibles recubridores universales.

Si el género de una superficie compacta es mayor o igual que 2, el recubridor universal es necesariamente el disco, cuyo grupo de automorfismos es isomorfo a $\mathrm{PSL}(2,\mathbb{R})$. Una de las principales características de este grupo de automorfismos es que coincide con el grupo de isometrías (que preservan la orientación) del disco con la métrica hiperbólica, y por lo tanto cualquier superficie de Riemann de género mayor o igual que dos hereda de forma natural una métrica hiperbólica. Los subgrupos de $\mathrm{PSL}(2,\mathbb{R})$ que definen una superficie de Riemann en el cociente no tienen porqué actuar libremente, basta con que actúen de manera propiamente discontinua. A tales grupos se les llama grupos fuchsianos.

Entre los grupos fuchsianos, una familia importante es la de los grupos triangulares, que son grupos generados por giros alrededor de los tres vértices de un triángulo hiperbólico y que definen en el cociente una superficie de Riemann de género 0 con tres puntos marcados. Los grupos triangulares están estrechamente relacionados con los dessins d'enfants, que son los objetos principales de estudio de este curso.

Un dessin d'enfant es un grafo finito bicoloreado en una superficie topológica compacta y orientable cuyo complementario es unión finita de discos topológicos. Todo dessin dota a la superficie topológica en la que está inmerso de una estructura de superficie de Riemann. Es más, por el Teorema de Belyi–Grothendieck esa superficie corresponde a una curva algebraica con coeficientes en el cuerpo de números algebraicos $\overline{\mathbb{Q}}$, y a la inversa, a toda curva con coeficientes algebraicos le corresponde al menos un dessin.

El objetivo de este minicurso es estudiar la existencia de múltiples dessins uniformes del mismo tipo en una superficie de Riemann. En el caso no aritmético se tiene un resultado inmediato, pero en el caso en el que el grupo que uniformiza la superficie es aritmético el estudio del número de dessins distintos guarda una estrecha relación con el de órdenes maximales en álgebras de cuaterniones. Gracias a ello, encontraremos una condición necesaria y suficiente para que una superficie de Riemann contenga varios dessins uniformes. También expondremos varios ejemplos de superficies de Riemann bien conocidas en las que, por los resultados anteriores, demostramos que viven varios dessins uniformes del mismo tipo.

Este curso está basado en un trabajo conjunto con Ernesto Girondo y Jürgen Wolfart:

- E. Girondo, D. Torres-Teigell, J. Wolfart: *Shimura curves with many uniform dessins*, Math. Z. (2011), doi:10.1007/s00209-011-0889-4.

Contents

CHAPTER 1

Introduction

In this chapter we introduce many basic notions which will be used later in the rest of the chapters.

In section 1.1 we give an introduction to Riemann surfaces. Although there is a huge amount of literature on this subject, perhaps the most suitable references for our purposes are [**19, 6, 8**].

Section 1.2 deals with Fuchsian groups and triangle groups, their fundamental domains and their relation with hyperbolic geometry. Most of what is presented here can be found in [**1, 13**].

In section 1.3 we present the Grothendieck–Belyi theory of dessins d'enfants and Belyi functions. We refer the reader to [**8**] for a comprehensive and more formal exposition (see also [**20**]).

Finally, in section 1.4 we begin the study of multiple dessins d'enfants on the same Riemann surface.

1.1. Riemann surfaces

A Riemann surface is a topological surface with a complex structure, i.e. with an atlas $\{(U_i, \varphi_i)\}$ such that the transition functions $\varphi_i \circ \varphi_j^{-1}$ are holomorphic functions between open sets of the complex plane $\mathbb{C}$. By the Cauchy–Riemann equations, every Riemann surface is orientable, and therefore the compact ones are topologically characterized by their genus.

The most basic examples of Riemann surfaces are open sets of the complex plane $U \subset \mathbb{C}$ with the identity atlas $\{(U, \mathrm{Id})\}$. In particular one has the complex plane $\mathbb{C}$, the upper half-plane $\mathbb{H} = \{\mathrm{w} \in \mathbb{C} \ : \ \mathrm{Im(w)} > 0\}$ and the unit disc $\mathbb{D} = \{\mathrm{w} \in \mathbb{C} \ : \ |\mathrm{w}| < 1\}$. Other surfaces that can be given a Riemann surface structure are the unit sphere $\mathbb{S}^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$, the complex extended plane (or Riemann sphere) $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ and the complex projective line $\mathbb{P}^1 := \mathbb{P}^1(\mathbb{C})$.

It is because of the complex structure that one can define in a natural way holomorphic and meromorphic functions on Riemann surfaces and morphisms between them. The following Riemann surfaces are isomorphic: $\mathbb{H} \cong \mathbb{D}$ and $\mathbb{S}^2 \cong \widehat{\mathbb{C}} \cong \mathbb{P}^1$. In fact, these two are, together with the complex plane $\mathbb{C}$ the only simply connected Riemann surfaces.

Theorem (Uniformisation theorem). *Any simply connected Riemann surface is isomorphic to $\mathbb{D}$, $\mathbb{C}$ or $\widehat{\mathbb{C}}$.*

It is a classical fact that there exists a bijection between isomorphism classes of compact Riemann surfaces and isomorphism classes of non-singular projective algebraic curves over the complex field. We will therefore use interchangeably the terms Riemann surface and algebraic curve.

As for automorphisms of compact Riemann surfaces, i.e. isomorphisms of S onto itself, there is a bound to the order of the automorphism group $\mathrm{Aut}(S)$ of S in terms of its genus $g(S)$. This bound, called Hurwitz bound, states that for $g(S) \geq 2$ one has $|\mathrm{Aut}(S)| \leq 84(g(S) - 1)$. The Riemann surfaces achieving it are called Hurwitz curves, and any finite group G which occurs as the full automorphism group of one of these surfaces is called a Hurwitz group.

The Galois group $\mathrm{Gal}(\mathbb{C}) := \mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ acts naturally on complex algebraic varieties in the following way. Let first $S = \{[x, y, z] \in \mathbb{P}^2(\mathbb{C}) : F(x, y, z) = 0\}$ be a projective algebraic curve given as the zeroes of a homogeneous polynomial $F \in \mathbb{C}[X, Y, Z]$. If $\sigma \in \mathrm{Gal}(\mathbb{C})$ is a field automorphism of $\mathbb{C}$ one can construct the Galois conjugate curve $S_F^\sigma = S_{F^\sigma}$, where F^σ is obtained from F by applying σ to its coefficients. We can proceed in the same way in higher dimension (or if the model for the curve S is not plane), so that if $V = \{F_\alpha = 0\}$ is an algebraic variety defined as the set of zeroes of a finite collection of polynomials $\{F_\alpha\} \subset \mathbb{C}[X_1, \ldots, X_n]$, the Galois conjugate variety is defined as the set of zeroes $V^\sigma = \{F_\alpha^\sigma = 0\}$.

Let now S be a compact Riemann surface and $k \subseteq \mathbb{C}$ a field. We say that k is a *field of definition* of S if there exists a finite collection of homogenous polynomials $F \subset k[X_1, \ldots, X_n]$ such that S and $S_F = \{[x_1, \ldots, x_n] \in \mathbb{P}^{n-1}(\mathbb{C}) : Q(x_1, \ldots, x_n) = 0, \text{ for all } Q \in F\}$ are isomorphic. On the other hand if we define the inertia group

$$I_S = \{\sigma \in \mathrm{Gal}(\mathbb{C}) : S_F^\sigma \cong S_F\},$$

which clearly does not depend on the algebraic model of S, then the fixed field

$$\mathbb{C}^{I_S} = \mathrm{Fix}(I_S) = \{\alpha \in \mathbb{C} : \sigma(\alpha) = \alpha, \text{ for all } \sigma \in I_S\}$$

is called the *field of moduli* of S, and it is denoted by $M(S)$. In particular the index of I_S in $\mathrm{Gal}(\mathbb{C})$ agrees with the cardinality of the orbit of S under the action of $\mathrm{Gal}(\mathbb{C})$. The field of moduli of a Riemann surface is always contained in any field of definition, but the converse is not true in general, as shown by well-known counterexamples ([**5, 21**]). The concepts of field of definition and field of moduli of a complex algebraic variety V of arbitrary dimension can be defined in the same way.

The general theory of covering spaces tells us that any topological manifold X admits a simply connected *universal covering* $\widetilde{X}$. Furthermore, if X has a complex structure the universal cover can be endowed with a complex structure such that the projection $\widetilde{X} \longrightarrow X$ is a morphism.

In the particular case of surfaces, this theory ensures that any Riemann surface S can be written as the quotient $S = \widetilde{S}/G$ of a simply connected Riemann surface $\widetilde{S}$ by the free action of a subgroup G of the group of automorphisms $\mathrm{Aut}(\widetilde{S})$, which is moreover isomorphic to the fundamental group $\pi_1(S)$. In this case, the situation is quite easy since by the uniformisation theorem the universal covering of any Riemann surface S must be isomorphic either to $\mathbb{D}$, $\mathbb{C}$ or $\mathbb{P}^1(\mathbb{C})$. Now the only Riemann surface having $\mathbb{P}^1(\mathbb{C})$ as universal cover is precisely $\mathbb{P}^1(\mathbb{C})$, since any automorphism of $\mathbb{P}^1(\mathbb{C})$ has fixed points. As for the complex plane, one has $\mathrm{Aut}(\mathbb{C}) \cong \{z \longmapsto az + b \ : \ a, b \in \mathbb{C}\}$ and any subgroup $G < \mathrm{Aut}(\mathbb{C})$ which does not fix points is a group of translations, therefore abelian; hence no compact Riemann surface of genus greater than or equal to two can have $\mathbb{C}$ as universal covering, since its fundamental group is not abelian. As a first consequence, it becomes particularly important the group of automorphisms of the disc, since almost every compact Riemann surface will be uniformised by a torsion-free subgroup of it.

The groups of automorphisms of $\mathbb{H}$ and $\mathbb{D}$ are isomorphic to $\mathrm{PSL}(2, \mathbb{R})$ and they can be identified with

$$\begin{aligned} \mathrm{Aut}(\mathbb{H}) &= \left\{ \mathrm{w} \longmapsto \frac{a\mathrm{w} + b}{c\mathrm{w} + d} \ : \ a, b, c, d \in \mathbb{R}, ad - bc \neq 0 \right\} \quad \text{and} \\ \mathrm{Aut}(\mathbb{D}) &= \left\{ \mathrm{w} \longmapsto e^{i\theta} \frac{\mathrm{w} - \alpha}{1 - \overline{\alpha}\mathrm{w}} \ : \ \alpha \in \mathbb{D}, \theta \in \mathbb{R} \right\}. \end{aligned}$$

Finally, the genus of a compact Riemann surface determines its universal covering.

Proposition 1.1. *Compact Riemann surfaces can be characterized in the following way:*

(i) *the only compact Riemann surface of genus zero is the Riemann sphere* $\mathbb{P}^1(\mathbb{C})$*;*
(ii) *the universal covering of any compact Riemann surface of genus one is the complex plane* $\mathbb{C}$*, and the group of deck transformations is a lattice:*

$$\Lambda = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2, \qquad \textit{with } \omega_1, \omega_2 \in \mathbb{C}, \textit{ and } \tfrac{\omega_1}{\omega_2} \notin \mathbb{R};$$

(iii) *the universal covering of any compact Riemann surface of genus greater than or equal to two is the upper half-plane* $\mathbb{H}$*, and the group of deck transformations is a subgroup* $\Gamma < \mathrm{PSL}(2, \mathbb{R})$.

1.2. Fuchsian groups. Triangle groups.

The subgroups of $\mathrm{Aut}(\mathbb{H})$ which define a Riemann surface structure on the quotient do not necessarily act without fixed points. A *Fuchsian group* is a subgroup $\Gamma < \mathrm{PSL}(2, \mathbb{R})$ which is discrete with respect to the topology induced by the usual topology in $\mathbb{R}^4$. Fuchsian groups were introduced by Henri Poincaré in 1880 following writings of Lazarus Fuchs about differential equations. One can prove that a subgroup $\Gamma < \mathrm{PSL}(2,\mathbb{R})$ is a Fuchsian group if and only if it acts discontinuously on $\mathbb{H}$, i.e.

(i) Every $\mathrm{w} \in \mathbb{H}$ is a fixed point of only a finite number of transformations $\gamma_1 = \mathrm{Id}, \dots, \gamma_r \in \Gamma$;
(ii) For every $\mathrm{w} \in \mathbb{H}$ there exists a neighbourhood U such that $\gamma(U) \cap U = \emptyset$ for every $\gamma \in \Gamma \setminus \{\gamma_1, \dots, \gamma_r\}$.

The quotient $\mathbb{H}/\Gamma$ of $\mathbb{H}$ by the action of a Fuchsian group Γ has a natural Riemann surface structure. The elements of Γ that fix points in $\mathbb{H}$ correspond precisely those of finite order. If the resulting Riemann surface $\mathbb{H}/\Gamma$ is compact, the set of conjugacy classes of finite order elements of Γ is finite. One can take suitable representatives γ_i of order m_i such that for every $\mathrm{w} \in \mathbb{H}$ the set of elements of Γ fixing it is either trivial or a cyclic group generated by an element conjugate to one of the γ_i. Under these assumptions, if the Riemann surface defined by Γ has genus g we say that Γ has *signature* $(g; m_1, \dots, m_k)$.

Let now $\Gamma, \Gamma' < \mathrm{PSL}(2,\mathbb{R})$ be Fuchsian groups acting without fixed points on $\mathbb{H}$ and $S = \mathbb{H}/\Gamma$ and $S' = \mathbb{H}/\Gamma'$ be the corresponding (not necessarily compact) Riemann surfaces uniformised by them. Then S and S' are isomorphic if and only if there exists $\gamma \in \mathrm{PSL}(2,\mathbb{R})$ such that $\Gamma' = \gamma\Gamma\gamma^{-1}$. Moreover $\mathrm{Aut}(\mathbb{H}/\Gamma) \cong N(\Gamma)/\Gamma$, where $N(\Gamma) = \{\gamma \in \mathrm{PSL}(2,\mathbb{R}) \ : \ \gamma\Gamma\gamma^{-1} = \Gamma\}$ is the normaliser of Γ in $\mathrm{PSL}(2,\mathbb{R})$.

If Γ is not cyclic then $N(\Gamma)$ is also a Fuchsian group, and therefore compact Riemann surfaces of genus greater than or equal to two have finite group of automorphisms.

One of the most relevant facts about holomorphic self-mappings of the disc is their relation with hyperbolic geometry. Let us first recall some concepts about this geometry. The basic idea behind hyperbolic (plane) geometry is replacing Euclid's fifth postulate (more precisely Playfair's axiom):

For any given line L and point P not on L, there is exactly one line through P that does not intersect L.

by the following one:

For any given line L and point P not on L, there are infinitely many lines through P that do not intersect L.

The *hyperbolic plane* satisfies this new axiom. It is a simply connected Riemannian manifold of dimension 2 whose metric has constant curvature -1. The metrics

$$ds^2_{\mathbb{H}} = \frac{dx^2 + dy^2}{y^2} \qquad \text{and} \qquad ds^2_{\mathbb{D}} = \frac{dx^2 + dy^2}{(1 - (x^2 + y^2))^2}$$

on the upper half-plane $\mathbb{H}$ and on the disc $\mathbb{D}$ respectively turn them into models of the hyperbolic plane. These metrics are conformal to the Euclidean one in $\mathbb{R}^2$, and therefore the Euclidean angles are preserved.

One can compute the hyperbolic length of a curve $\gamma(t) = (x(t), y(t))$ and the hyperbolic area of a set E contained in $\mathbb{H}$ or in $\mathbb{D}$ through the formulae

$$\ell_{\mathbb{H}}(\gamma) = \int \frac{\sqrt{x'(t)^2 + y'(t)^2}}{y(t)} dt\,, \qquad A_{\mathbb{H}}(E) = \int\int_E \frac{dxdy}{y^2}\,,$$

$$\ell_{\mathbb{D}}(\gamma) = \int \frac{\sqrt{x'(t)^2 + y'(t)^2}}{1 - (x(t)^2 + y(t)^2)} dt\,, \qquad A_{\mathbb{D}}(E) = \int\int_E \frac{dxdy}{(1 - (x^2 + y^2))^2}\,.$$

In both models the geodesics of the hyperbolic metric are arcs of (generalised) circumferences which intersect perpendicularly the border, $\partial\mathbb{H} = \mathbb{R} \cup \{\infty\}$ in the case of $\mathbb{H}$ and $\partial\mathbb{D} = \mathbb{S}^1$ in the case of $\mathbb{D}$.

The group $\mathrm{Aut}(\mathbb{H}) \cong \mathrm{PSL}(2,\mathbb{R})$ of holomorphic self-mappings of $\mathbb{H}$ coincides with the group of orientation-preserving isometries of the hyperbolic metric and acts transitively on the set of hyperbolic geodesics. In particular, its elements preserve both hyperbolic distance and hyperbolic area.

Let us now consider a Fuchsian group $\Gamma < \mathrm{PSL}(2,\mathbb{R})$ acting on the upper-half plane. We will call *fundamental domain* of Γ to any closed subset $\Omega \subset \mathbb{H}$ such that:

(i) Ω contains at least one point of each orbit of Γ;

(ii) the interior of Ω does not contain points equivalent under Γ;
(iii) $A_{\mathbb{H}}(\partial\Omega) = 0$, where $\partial\Omega$ is the border of Ω.

If Ω is a fundamental domain, $\mathbb{H} = \bigcup_{\gamma\in\Gamma}\gamma(\Omega)$ and we say that Ω and its images under Γ form a *tessellation* of $\mathbb{H}$. There is a specific kind of fundamental domains with particularly nice properties. Let p be a point not fixed by any non-trivial element of Γ. We call *Dirichlet region* of Γ centered at p to the set

$$D_p(\Gamma) = \{\mathrm{w} \in \mathbb{H} \ : \ \rho_{\mathbb{H}}(\mathrm{w}, p) \leq \rho_{\mathbb{H}}(\gamma(\mathrm{w}), p), \ \forall \gamma \in \Gamma\},$$

where $\rho_{\mathbb{H}}$ is the hyperbolic distance. The region $D_p(\Gamma)$ is an intersection of hyperbolic half-planes and therefore it is a convex hyperbolic polygon, i.e. a closed connected set on $\overline{\mathbb{H}}$ whose border is formed by arcs of hyperbolic geodesics. As a consequence one can represent the compact Riemann surface $\mathbb{H}/\Gamma$ as a fundamental polygon $\mathcal{P}$ together with a side pairing on the sides $s_1, \ldots, s_n$, so that for every s_i there is an $s_{j(i)}$ and a $\gamma \in \Gamma$ such that $\gamma(s_i) = s_{j(i)}$.

Moreover, if the group of elements of Γ fixing a vertex $v_j \in \mathcal{P}$ is generated by $\gamma_j \in \Gamma$, then the angle at v_j is $\alpha_j = 2\pi/\mathrm{ord}(\gamma_j)$. The converse is included in the following theorem.

THEOREM 1.1 (Poincaré). *Let $\mathcal{P} \subset \overline{\mathbb{H}}$ be a hyperbolic polygon with (not necessarily ordered) sides $s_1, \ldots, s_n$, $s'_1, \ldots, s'_n$. Suppose that there exist elements $\gamma_i \in \mathrm{PSL}(2,\mathbb{R})$ such that $\gamma_i(s_i) = s'_i$ for each $i = 1, \ldots, n$ and let $\Gamma = \langle \gamma_1, \ldots, \gamma_n \rangle$. If for any complete collection V_j of vertices of $\mathcal{P}$ equivalent under Γ the sum of its angles is equal to $2\pi/m_j$ with $m_j \in \mathbb{N}$, then the group Γ acts properly discontinuously on $\mathbb{H}$ and $\mathbb{H}/\Gamma$ is a Riemann surface. If moreover $\mathcal{P} \cap \partial\mathbb{H} = \emptyset$, then $\mathbb{H}/\Gamma$ is compact.*

A special class of Fuchsian groups is that of Triangle groups. Let l, m and n be integers such that $1/l + 1/m + 1/n < 1$. To construct a triangle group of signature (l, m, n) one considers a hyperbolic triangle T in the hyperbolic plane, with vertices v_0, v_1 and v_∞ and angles π/l, π/m and π/n respectively. The reflection R_i over the edge of T opposite to v_i is an anticonformal isometry of the hyperbolic plane. The group generated by these reflections acts discontinuously on $\mathbb{H}$ in such a way that T is a fundamental domain. The index-2 subgroup formed by the orientation-preserving transformations is called a triangle group of type (l, m, n). Elementary hyperbolic geometry ensures that the triangle T, and hence the corresponding triangle group, that will be denoted by $\Delta = \Delta(l, m, n)$, are unique up to conjugation in $\mathrm{PSL}(2,\mathbb{R})$.

The quadrilateral consisting of the union of T and one of its reflections $R_i(T)$ (e.g. the shaded triangle in the figure) serves as a fundamental domain for the group $\Delta(l, m, n)$, and therefore its images under $\Delta(l, m, n)$ tessellate the whole hyperbolic plane. Thus, the quotient $\mathbb{H}/\Delta$ is an orbifold of genus zero with three cone points $[v_0]_\Delta$, $[v_1]_\Delta$ and $[v_\infty]_\Delta$ of orders l, m and n respectively, where for an arbitrary Fuchsian group Λ the notation $[v]_\Lambda$ stands for the orbit of the point v under the action of Λ.

It is a classical fact that $\Delta(l, m, n)$ has presentation

$$\Delta(l, m, n) = \langle x, y, z \ : \ x^l = y^m = z^n = xyz = 1 \rangle\,,$$

where

$$x = R_1 R_\infty\,, \quad y = R_\infty R_0\,, \quad z = R_0 R_1\,, \tag{1.1}$$

are positive rotations around v_0, v_1 and v_∞ through angles $2\pi/l$, $2\pi/m$ and $2\pi/n$ respectively. It is also classical that any other finite order element of $\Delta(l, m, n)$ is conjugate to a power of x, y or z and that these account for all elements in Δ that fix points. We will always identify $\mathbb{H}/\Delta$ with $\mathbb{P}^1$ via the unique isomorphism

$$\begin{array}{rccc} \Phi : & \mathbb{H}/\Delta & \longrightarrow & \mathbb{P}^1 \\ & [v_0]_\Delta & \longmapsto & 0 \\ & [v_1]_\Delta & \longmapsto & 1 \\ & [v_\infty]_\Delta & \longmapsto & \infty \end{array} \tag{1.2}$$

These groups are rigid among Fuchsian groups, in the sense that the quotient orbifold $\mathbb{H}/\Delta(l, m, n)$ does not admit non-trivial deformations.

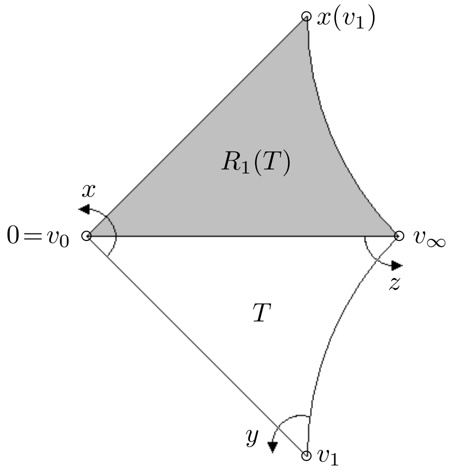


FIGURE 1.1. Generators x, y and z together with a fundamental domain of $\Delta(l, m, n)$ (depicted inside the unit disc model of the hyperbolic plane).

It is a well-known fact (see [**23**]) that the normaliser $N(\Delta)$ in $\mathrm{PSL}(2, \mathbb{R})$ of a triangle group $\Delta \equiv \Delta(l, m, n)$ is a triangle group again, and that the quotient $N(\Delta)/\Delta$ is faithfully represented in the symmetric group $\mathfrak{S}_3$ via its action on the vertices $[v_0]$, $[v_1]$, $[v_\infty]$ of the orbifold $\mathbb{H}/\Delta$. Thus

$$N(\Delta)/\Delta \cong \begin{cases} \{1\}, & \text{if } l,\ m \text{ and } n \text{ are all distinct;} \\ \mathfrak{S}_2, & \text{if } l = m \neq n; \\ \mathfrak{S}_3, & \text{if } l = m = n. \end{cases} \tag{1.3}$$

where $\mathfrak{S}_k$ stands for the symmetric group on k elements.

In the second case, a representative for the non-trivial element $(1, 2) \in \mathfrak{S}_2$ is the rotation $\lambda_4 \in N(\Delta)$ of order two around the midpoint of the segment joining v_0 and v_1 (see Figure 1.2). Conjugation by this element yields an order two automorphism of Δ which interchanges x and y and sends z to $x^{-1}zx$. We will denote it by $\tilde{\sigma}_4$.

In the case when $l = m = n$ we can choose the same representative λ_4 for the element $(1, 2) \in \mathfrak{S}_3$, and the order three rotation λ_1 in the positive sense around the incentre of T (i.e. the point where the three angle bisectors meet, see [**1**] §7.14) for $(1, 2, 3) \in \mathfrak{S}_3$. Conjugation by the latter induces an automorphism $\tilde{\sigma}_1$ of Δ of order three which sends x to y and y to z (see Figure 1.2).

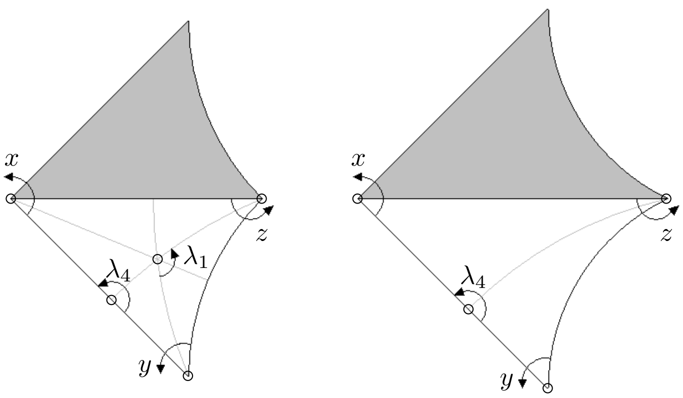


FIGURE 1.2. Generators of $\Delta(l, l, l)$ and $\Delta(l, l, n)$, and representatives of $(1, 2), (1, 2, 3) \in \mathfrak{S}_3$.

It is worth noting that in the case when $N(\Delta)/\Delta = \mathfrak{S}_2$ or $\{1\}$ the extension splits, i.e $N(\Delta) = \Delta \times (N(\Delta)/\Delta)$, but when $N(\Delta)/\Delta = \mathfrak{S}_3$ it does not, since no Fuchsian group can contain a noncyclic finite group. This means that the representatives of $N(\Delta)/\Delta$ cannot be chosen naturally to form a complement of Δ.

To summarize, $N(\Delta)$ can be written as

$$N(\Delta) \cong \begin{cases} \Delta, & \text{if } l,\, m \text{ and } n \text{ are all distinct;} \\ \langle\, \Delta, \lambda_4 \,\rangle, & \text{if } l = m \neq n; \\ \langle\, \Delta, \lambda_1, \lambda_4 \,\rangle, & \text{if } l = m = n. \end{cases} \tag{1.4}$$

1.3. Dessins d'enfants and Belyi functions

In the Grothendieck-Belyi theory of dessins d'enfants there are two main ingredients. First, a *dessin d'enfant* is a pair $(S, \mathcal{D})$, where S is a compact oriented topological surface and $\mathcal{D}$ is a finite graph embedded in S satisfying the following properties:

(i) it is a bicoloured graph, i.e. every vertex has an assigned colour, white ($\circ$) or black ($\bullet$), in such a way that the two vertices of an edge have always different colours;

(ii) each connected component of the complement $S \setminus \mathcal{D}$ is homeomorphic to a disc. Each of them will be called face of the dessin.

We will regard two dessins $(S_1, \mathcal{D}_1)$ and $(S_2, \mathcal{D}_2)$ as equivalent (or isomorphic) if there exists an orientation-preserving homeomorphism $f : S_1 \longrightarrow S_2$ whose restriction $f|_{\mathcal{D}_1}$ induces an isomorphism of bicoloured graphs $f|_{\mathcal{D}_1} : \mathcal{D}_1 \longrightarrow \mathcal{D}_2$. The degree of a vertex of $\mathcal{D}$ is defined as the number of incident edges and the degree of a face is defined as half the number of edges delimiting that face, counting multiplicities. If the least common multiples of the degrees of the white vertices, black vertices and faces are l, m and n respectively, we will say that the type of the dessin is (l, m, n).

The other ingredient is Belyi functions. A *Belyi function* is a meromorphic function $\beta : S \longrightarrow \mathbb{P}^1$ on a Riemann surface S with three ramification values at most, which we can suppose to be 0, 1 and ∞. We will consider two Belyi pairs (S, f) and (S', f') equivalent if there exists an isomorphism $F : S \longrightarrow S'$ such that $f = f' \circ F$.

Grothendieck pointed out that there is a bijective correspondence between equivalence classes of dessins d'enfants and equivalence classes of Belyi pairs. To recover a dessin from a Belyi function β one simply takes the inverse image of the interval $[0, 1]$ under β and considers $\beta^{-1}(0)$ as white vertices and $\beta^{-1}(1)$ as black vertices. Constructing a Belyi function from a dessin $\mathcal{D}$ is slightly more complicated. It can be achieved by considering a triangulation associated to $\mathcal{D}$ and constructing a topological covering β from S minus the set of vertices and face centres of $\mathcal{D}$ to $\mathbb{P}^1 \setminus \{0, 1, \infty\}$, which endows S with a Riemann surface structure $S_{\mathcal{D}}$ to which β extends as a meromorphic function with three ramification values. The degree of a given white vertex, black vertex or face of the dessin can be understood then as the ramification order of β in such point.

The importance of this fact lies on its relation with the theorem of Belyi ([**2**]), which states that a compact Riemann surface S is isomorphic to an algebraic curve defined over the field of algebraic numbers $\overline{\mathbb{Q}}$ if and only if there exists a Belyi function $f : S \longrightarrow \mathbb{P}^1$.

The fact that any Riemann surface admitting a Belyi function can be defined over $\overline{\mathbb{Q}}$ was already known and it follows from Weil's criterion ([**28**], see also [**11**]). However the proof of the other implication, which is due to Belyi, is as astonishing as simple. Grothendieck himself wrote about it in [**12**]: *"[...]Belyi annonce justement ce résultat, avec une démonstration d'une simplicité déconcertante tenant en deux petites pages d'une lettre de Deligne – jamais sans doute un résultat profond et déroutant ne fut démontré en si peu de lignes!"* [1]. This proof is based on constructing a function f from S to the sphere $\mathbb{P}^1$ ramified only over rational values, and compose it with suitable Belyi polynomials, which are polynomials of the form

$$P_{m,n}(\mathrm{w}) = \frac{(m+n)^{m+n}}{m^m \cdot n^n} \mathrm{w}^m (1 - \mathrm{w})^n.$$

The relevant fact is that 0, 1, $\frac{m}{m+n}$ and ∞ are the only ramification points of this polynomial, and they are sent to $\{0, 1, \infty\}$. Therefore, one can compose the function f with consecutive suitable polynomials P_{m_i,n_i} so that so that the set of ramification values of the resulting function ends up being the set $\{0, 1, \infty\}$.

[1] The translation into English that can be found in the introduction of [**20**] reads: *"[...]Belyi announced exactly that result, with a proof of a disconcerting simplicity which fit into two little pages of a letter of Deligne – never, without a doubt, was such a deep and disconcerting result proved in so few lines!".*

THEOREM (Belyi–Grothendieck). *For any Riemann surface S of genus g defined over $\overline{\mathbb{Q}}$ there exists a dessin $\mathcal{D}$ on the compact oriented topological surface of genus g such that $S = S_{\mathcal{D}}$.*

The importance of triangle groups in Grothendieck's theory of dessins d'enfants comes from the fact that any Belyi function β in a Riemann surface S can be represented as the natural projection $\mathbb{H}/\Lambda \longrightarrow \mathbb{H}/\Delta(l,m,n)$ from the quotient surface $\mathbb{H}/\Lambda \cong S$ to an orbifold $\mathbb{H}/\Delta(l,m,n)$ given by the inclusion $\Lambda < \Delta(l,m,n)$, where the signature of $\Delta(l,m,n)$ depends on the ramification orders of β ([**3, 29**]).

We have two important families of dessins. A dessin d'enfant $\mathcal{D}$ of type (l,m,n) (and its associated Belyi function) on a Riemann surface S is called *uniform* if all white vertices, black vertices and faces have degree l, m and n respectively. In the specific case where β is a uniform Belyi function of type (l,m,n), it corresponds to the inclusion of a torsion-free group K in the triangle group $\Delta(l,m,n)$. The group K is, of course, isomorphic to the fundamental group $\pi_1(S)$.

If, moreover, the automorphism group $\mathrm{Aut}(S)$ acts transitively on the edges of the dessin, we say that $\mathcal{D}$ is *regular*. A regular Belyi function corresponds to the normal inclusion of a uniformising group K of S in $\Delta(l,m,n)$, so that $\mathbb{H}/K \longrightarrow \mathbb{H}/\Delta(l,m,n)$ is a Galois covering with group $G \cong \Delta(l,m,n)/K$. Riemann surfaces which admit a regular Belyi function are called *quasiplatonic curves* (or triangle curves). In the next section we will make this connection between quasiplatonic curves and their covering groups G more explicit.

In these two cases one can study renormalisations of the dessin. Suppose that $\mathcal{D}$ is a uniform dessin on a Riemann surface S associated to a Belyi function β, and suppose that some of the orders of its type (l,m,n) are repeated. Then one can construct other dessins on the same surface by renormalisation in the following way. Consider an automorphism $F : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ of the Riemann sphere which permutes the ramification values of β of the same order, i.e. F permutes, for instance, 0 and 1 if $l = m \neq n$, and F permutes 0, 1 and ∞ if $l = m = n$. In this way the map $\beta_F = F \circ \beta$ is a Belyi function again, and the corresponding dessin $\mathcal{D}_F$ is called a renormalised dessin of $\mathcal{D}$.

Now, if the original Belyi function was given by an inclusion $K < \Delta$, the renormalised function β_F is induced by an element of $N(\Delta)$ in the following way: there exists $\alpha \in N(\Delta)$ whose action on v_0, v_1 and v_∞ coincides with the action of F on 0, 1 and ∞, and then β_F is given by the inclusion $\alpha K \alpha^{-1} < \Delta$, since one has

$$\beta_F : \mathbb{H}/K \xrightarrow{\ \beta\ } \mathbb{H}/\Delta \xrightarrow{\ \alpha\ } \mathbb{H}/\Delta$$

In the case when α additionally belongs to $N(K)$, there exists an isomorphism $\phi \in \mathrm{Aut}(S)$ such that $\beta \circ \phi = \beta_F$, and $\mathcal{D}_F$ and $\mathcal{D}$ are isomorphic.

1.4. Multiple dessins d'enfants on a surface

The correspondence between (equivalence classes of) dessins d'enfants and (isomorphism classes of) algebraic curves defined over $\overline{\mathbb{Q}}$ is not bijective, given a Riemann surface defined over $\overline{\mathbb{Q}}$, there are many different dessins d'enfants on S. However, the question of when two different dessins d'enfants live on the same surface is too wide to answer in its full generality, so one has to restrict to certain families of dessins.

In [**10**] it was considered the case of regular dessins of the same type (see also [**7**]). Let us remind that a regular dessin of type (l,m,n) on a surface S arises as the normal inclusion of a group K uniformising S in a triangle group $\Delta(l,m,n)$ and, therefore, the situation of several regular dessins of the same type on S corresponds to the normal inclusion of K in different conjugate triangle groups of type (l,m,n). Girondo and Wolfart proved that if this happens, these inclusions are induced by inclusions between triangle groups.

The next family of dessins that one could study is that of uniform dessins. Recall that a uniform dessin of type (l,m,n) on a surface S arises as the inclusion – not necessarily normal – of a group K uniformising S in a triangle group $\Delta(l,m,n)$. As a consequence, the existence of several uniform dessins of type (l,m,n) corresponds to the inclusion of K in different triangle groups of type (l,m,n)

To put the problem in a precise form we observe first that a surface group K contained in a triangle group Δ is contained in all triangle groups Δ' containing Δ (and maybe also in some triangle subgroups of Δ), all these inclusions inducing dessins of different types on the surface S. All possibilities of such inclusions are well known by work of Singerman [**23**], so one can concentrate on dessins of the same type (l, m, n), i.e. on the following question. Let K be a Fuchsian surface group contained in a triangle group $\Delta(l, m, n)$: which and how many different conjugate groups $\alpha\Delta\alpha^{-1}$, $\alpha \in \mathrm{PSL}(2, \mathbb{R})$, contain K as well?

Note that one could consider the following similar question. Let Δ be a Fuchsian triangle group and let K be a finite index subgroup: for which and for how many $\alpha \in \mathrm{PSL}(2, \mathbb{R})$ do we have $\alpha^{-1}K\alpha < \Delta$?

These two questions are not equivalent only when α belongs either to $N(\Delta)$ or to $N(K)$, the normalisers of Δ and K in $\mathrm{PSL}(2, \mathbb{R})$. However, if $\alpha \in N(\Delta)$ the two inclusions $K, \alpha^{-1}K\alpha < \Delta$ correspond to renormalised dessins, and if $\alpha \in N(K)$, conjugation by α induces an isomorphism of the curve, which in turn induces an isomorphism between the dessins corresponding to $K < \Delta$ and $K < \alpha\Delta\alpha^{-1}$. Therefore, if one wants to study the problem of non-isomorphic dessins of the same type on the same surface not related by renormalisation, then both questions are interchangeable. We will thus focus on the first version, which is more natural. Moreover, since conjugation of K by an element of $N(\Delta)$ induces a renormalisation of the dessin, we will only count residue classes $\alpha \in \mathrm{PSL}(2, \mathbb{R})/N(\Delta)$. Let us stress here that, in view of the question we are dealing with, two dessins are considered different if they correspond to different Belyi functions. Whether they are isomorphic or not is a completely different question.

If K is included in both Δ and $\alpha\Delta\alpha^{-1}$, the element α belongs by definition to both the commensurator groups of K and of Δ. Now, by the theorem of Margulis (see section 3.1) the commensurator $\overline{\Delta} = \mathrm{Comm}(\Delta)$ of a non-arithmetic Fuchsian group Δ is a finite extension of Δ and a Fuchsian group itself. But finite extensions of triangle groups are known to be triangle groups again, so if Δ is non-arithmetic, $\overline{\Delta}$ is itself a triangle group, and consulting Takeuchi's list of arithmetic triangle groups [**26**] and Singerman's list of inclusion relations [**23**] it is easy to see that the index $[\overline{\Delta} : \Delta]$ is at most 6. So we have the first part of the following theorem.

THEOREM 1.2. *Surface groups contained in a non-arithmetic Fuchsian triangle group* Δ *define isomorphic surfaces if and only if they are conjugate in a maximal Fuchsian triangle group* $\overline{\Delta}$ *extending* Δ. *They fall in at most* 6 *different conjugacy classes under conjugation by* Δ. *If* K *is such a surface group then the number of triangle groups conjugate to* Δ *in which* K *is included is* 1, 3 *or* 4.

PROOF. The second part of the theorem follows from the fact that non-normal inclusions $\Delta < \overline{\Delta}$ of non-arithmetic triangle groups occur only with index 3 for $\Delta(2, n, 2n) < \Delta(2, 3, 2n)$, or 4 for $\Delta(3, n, 3n) < \Delta(2, 3, 3n)$. □

CHAPTER 2

Quaternion algebras

For an exhaustive introduction on the theory of quaternion algebras see for example [**27, 17**] (see also [**16, 14**]). Much of what is written here is taken from those books.

We will also need some basic notions on field theory and field completions. All the preliminaries needed can be found in [**17**] for instance.

2.1. Field completions. p-adic fields

Let k be a number field. A valuation v on k is a map $v : k \longrightarrow \mathbb{R}^+$ satisfying the following three properties:

(i) $v(x) = 0$ if and only if $x = 0$.
(ii) v is a homomorphism of (multiplicative) groups, i.e. $v(xy) = v(x)v(y)$ for all $x, y \in k$.
(iii) $v(x + y) \leq v(x) + v(y)$ for all $x, y \in k$.

If additionally $v(x+y) \leq \max\{v(x), v(y)\}$ for all $x, y \in k$ we say that v is a non-Archimedean valuation. Otherwise it is Archimedean.

We will always suppose that v is not the trivial valuation, i.e. v is not identically 1 on k^*. Two valuations v and v' are equivalent if there exists $a \in \mathbb{R}^+$ such that $v'(x) = v(x)^a$ for all $x \in k$. We call an equivalence class of valuations a *place* of k and denote by $\Omega(k)$ the set of all places of k.

All Archimedean valuations correspond to $v_\sigma(x) = |\sigma(x)|$ for $x \in k$, where $\sigma : k \to \mathbb{C}$ is a Galois embedding of k in $\mathbb{C}$ and $|\cdot|$ stands for the usual norm in $\mathbb{C}$. Moreover, two such valuations v_σ and $v_{\sigma'}$ are equivalent if and only if σ and σ' are complex conjugate embeddings.

Let now v be a non-Archimedean valuation. The valuation ring

$$R(v) = \{x \in k : v(x) \leq 1\}$$

is a local ring with maximal ideal

$$P(v) = \{x \in k : v(x) < 1\}$$

and whose field of fractions is precisely k.

Write R_k for the ring of integers of k. For any prime ideal $\mathfrak{p}$ in R_k, write $v_\mathfrak{p}(x) = N(\mathfrak{p})^{-n_\mathfrak{p}(x)}$ for $x \in R_k \backslash \{0\}$, where $N(\mathfrak{p})$ stands for its norm and $n_\mathfrak{p}(x)$ is the largest integer m such that $x \in \mathfrak{p}^m$. This definition can be extended to k^* by the rule $v_\mathfrak{p}(x/y) = v_\mathfrak{p}(x)/v_\mathfrak{p}(y)$. This defines a non-Archimedean valuation, and all non-Archimedean valuations are equivalent to some $v_\mathfrak{p}$.

If $[k : \mathbb{Q}] = d$, we can write $d = r_1 + 2r_2$, where r_1 is the number of real embeddings of k and r_2 is the number of complex conjugate pairs of complex embeddings of k. Therefore there are $r_1 + r_2$ Archimedean places on k, which are called *infinite places*, in contrast to non-Archimedean places, which are called *finite places*. We will write $\Omega_\infty(k)$ for the set consisting of the former, and $\Omega_f(k)$ for the one consisting of the latter.

For non-Archimedean valuations, any element π such that $n_\mathfrak{p}(\pi) = 1$ is called a *uniformiser*. One has that $P(v_\mathfrak{p}) = \pi R(v_\mathfrak{p})$, that is π generates the maximal ideal of $R(v_\mathfrak{p})$. Moreover, $R(v_\mathfrak{p})$ is a principal ideal domain whose ideals are of the form $\pi^n R(v_\mathfrak{p})$. The quotient fields $R(v_\mathfrak{p})/P(v_\mathfrak{p})$ and $R_k/\mathfrak{p}$ are isomorphic finite fields of order $N(\mathfrak{p})$ and they are called the *residue field* of $\mathfrak{p}$.

Now note that, given a valuation v, the formula $d_v(x, y) = v(x - y)$ defines a metric on k. The field k is not in general a complete metric space. However, for each valuation v one can consider its completion k_v, which is uniquely determined up to isomorphism. One always have an inclusion $i_v(k)$ of k inside k_v. Moreover, Archimedean (resp. non-Archimedean) valuations v on k extend to Archimedean (resp. non-Archimedean) valuations $\hat{v}$ on k_v.

It is known that the only complete fields with Archimedean valuations are $\mathbb{R}$ and $\mathbb{C}$, and that such a valuation must be equivalent to the usual absolute value. As a consequence, for each infinite place v on k one has $k_v = \mathbb{R}$ or $\mathbb{C}$.

If $v = v_\mathfrak{p}$ is a non-Archimedean valuation, we write $k_\mathfrak{p} = k_v$ for the completion of k with respect to v. The valuation ring $R_\mathfrak{p}$ of the extended valuation $\hat{v}_\mathfrak{p}$ on $k_\mathfrak{p}$ is called the *ring of* $\mathfrak{p}$*-adic integers*, and it is again a local ring with maximal ideal generated by $\pi \equiv i_v(\pi)$. Moreover, the residue fields $R_\mathfrak{p}/\pi R_\mathfrak{p}$ and $R(v_\mathfrak{p})/\pi R(v_\mathfrak{p})$ are isomorphic.

Finally, one can prove that every element $\alpha \in k_\mathfrak{p}$ can be written uniquely as a power series

$$\alpha = \pi^r \sum_{n=0}^{\infty} a_n \pi^n ,$$

where $r \in \mathbb{Z}$, $a_n \in R_\mathfrak{p}/\pi R_\mathfrak{p}$ and $a_0 \neq 0$.

2.2. Basic properties of quaternion algebras

The following definition works for any field k, but we will only focus on subfields of the complex field and its localisations (some of the following statements might not be true in characteristic 2). A *quaternion algebra* A over k is a 4-dimensional central simple $k-$algebra, i.e. a $k-$algebra of dimension 4 without proper two-sided ideals (in the sense of ring theory), whose centre agrees with the field k. There always exist $a, b \in k^*$ and a basis $\{1, i, j, ij\}$ of A such that we can write

$$A = \{x_0 + x_1 i + x_2 j + x_3 ij : x_0, x_1, x_2, x_3 \in k, i^2 = a, j^2 = b, ij = -ji\}.$$

Note that $(ij)^2 = -ab$. Conversely, any choice of $a, b \in k^*$ defines a quaternion algebra A over k. Under these conditions we will denote it by the Hilbert symbol $A = (\frac{a,b}{k})$. However different choices of a and b can lead to isomorphic algebras.

In fact, it is easy to see that, for each $a, b, x, y \in k^*$ one has

$$\left(\frac{a,b}{k}\right) \cong \left(\frac{b,a}{k}\right) \cong \left(\frac{a,-ab}{k}\right) \cong \left(\frac{-ab,b}{k}\right) \cong \left(\frac{ax^2,by^2}{k}\right) .$$

Given an element $x = x_0 + x_1 i + x_2 j + x_3 ij$, its conjugate is defined as $\overline{x} = x_0 - x_1 i - x_2 j - x_3 ij$. This allows us to define a *reduced norm* and a *reduced trace* on A as

$$n(x) = x\overline{x} = x_0^2 - ax_1^2 - bx_2^2 + abx_3^2, \quad \text{and} \quad \mathrm{tr}(x) = x + \overline{x} = 2x_0.$$

All these definitions do not depend on the choice of basis. The invertible elements of A, the set of which is denoted by A^*, are precisely those x such that $n(x) \neq 0$. We will write $A^1 \subset A^*$ for the subgroup of elements of norm 1.

It is known that any quaternion $k-$algebra A is either a division algebra or isomorphic to $M_2(k)$. In the case of k being algebraically closed, A is necessarily isomorphic to $M_2(k)$ and therefore there is only one quaternion k-algebra. Division quaternion algebras are characterized by the fact that $x = 0$ is the only element with norm zero.

Now write t for $\sqrt{a}$. For any quaternion k-algebra $A = (\frac{a,b}{k})$ one can consider the linear map

$$\begin{array}{rccl} \rho: & A & \longrightarrow & M_2(k(t)) \\ & 1 & \longmapsto & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ & i & \longmapsto & \begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix} \\ & j & \longmapsto & \begin{pmatrix} 0 & 1 \\ b & 0 \end{pmatrix} \\ & ij & \longmapsto & \begin{pmatrix} 0 & t \\ -bt & 0 \end{pmatrix} \end{array} \tag{2.1}$$

from A into the 2×2 matrices over the quadratic field extension $k(t)$ of k. It is easy to check that this map determines an isomorphism of k-algebras between A and $\rho(A)$, so one can always regard any quaternion algebra as an algebra of matrices. Moreover, via this identification the reduced norm $n(x)$ and the reduced trace $\mathrm{tr}(x)$ on A coincide with the matrix determinant $\det(\rho(x))$ and the matrix trace $\mathrm{tr}(\rho(x))$.

If $a = x^2$ for some $x \in k$, then ρ is an isomorphism between A and $M_2(k)$. As a consequence of this fact and of the previous equivalences of Hilbert symbols, for each $b, x \in k^*$ one has

$$\left(\frac{x^2,b}{k}\right) \cong \left(\frac{b,x^2}{k}\right) \cong \left(\frac{b,-b}{k}\right) \cong M_2(k) .$$

2.2.1. Subalgebras, ideals and orders. A k-subalgebra of a k-algebra A is simply a subset of A which is closed under all the induced operations.

The following theorem is a crucial result on central simple algebras.

THEOREM (Skolem–Noether). *Let A, B be finite-dimensional central simple k-algebras. For any two homomorphisms of algebras $\phi, \psi : B \longrightarrow A$, there exists an invertible element $c \in A$ such that $\phi(x) = c\psi(x)c^{-1}$ for all $x \in B$.*

In particular one has that every endomorphism of a quaternion algebra is in fact an inner automorphism. Another consequence of the Skolem–Noether theorem is that any isomorphism between subalgebras of a quaternion algebra is induced by an inner automorphism of A ([**27**], Ch. I, Thm 2.1).

We will be interested in certain discrete subsets of quaternion algebras called orders, which are special cases of other structures called ideals (not to be confused with ring ideals).

First of all, let R_k be the ring of integers of k. Given any k-vector space V, an *R_k-lattice* L over V is a finitely generated R_k-module contained in V. We say that L is complete if $L \otimes_{R_k} k = V$, that is when extending scalars to generate a k-module one gets the whole vector space. A complete R_k-lattice in a quaternion algebra is also called an *ideal*.

One has the following characterization of complete lattices.

LEMMA 2.1. *Let L and M be R_k-lattices in V. Then M is complete if and only if there exists an integer $a \in R_k$ such that $aL \subset M \subset a^{-1}L$.*

One can generalise integers in fields to quaternion algebras in the following way. An element α in a quaternion algebra A is an *integer* if $R_k[\alpha]$ is an R_k-lattice in A. Integers in a quaternion algebra A are characterised as the elements α whose reduced norm $n(\alpha)$ and reduced trace $\mathrm{tr}(\alpha)$ both lie in R_k.

It must be noted that integers in quaternion algebras do not behave exactly as integers in fields. For example, the sum and product of integers is not necessarily an integer. Orders in quaternion algebras are an analogue of the ring of integers on a field.

An *order* $\mathcal{O}$ in a quaternion algebra A is an ideal which is also a ring with unity. Equivalently, one can define an order $\mathcal{O}$ in A as a ring of integers which contains R_k and such that $k\mathcal{O} = A$.

Every order is contained in a *maximal order*, that is an order which is maximal with respect to the inclusion. We call $\mathcal{O}$ an *Eichler order* if it is the intersection of two maximal orders ([**27**], p. 20).

In the case where $A = M_2(k)$ and R_k is a principal ideal domain, all maximal orders are conjugate in A to $M_2(R_k)$. In general the number of conjugacy classes of maximal orders of A is called the *type number* of A.

2.3. Algebras over local fields. Algebras over global fields

The easiest examples of quaternion algebras are the division algebra of Hamilton's quaternions $\mathcal{H} = (\frac{-1,-1}{\mathbb{R}})$ and the algebra of matrices $M_2(\mathbb{R}) = (\frac{1,1}{\mathbb{R}})$. Note first that $(\frac{1,-1}{\mathbb{R}}) = (\frac{-1,1}{\mathbb{R}}) = M_2(\mathbb{R})$.

Now for every $a, b \in \mathbb{R} \setminus \{0\}$ write $a = \pm x^2$ and $b = \pm y^2$, and therefore one has that $(\frac{a,b}{\mathbb{R}}) = (\frac{\pm 1,\pm 1}{\mathbb{R}})$, which will be equal to $\mathcal{H}$ or $M_2(\mathbb{R})$ depending on whether both a and b are negative or not, respectively. As a consequence these two are the only quaternion algebras over the real field.

An analogous situation occurs over p-adic fields.

THEOREM 2.1. *Let k be a number field and $v = v_{\mathfrak{p}}$ a non-Archimedean valuation on k corresponding to the prime ideal $\mathfrak{p} \in R_k$. The only quaternion algebras over the field $k_{\mathfrak{p}}$ are the algebra of matrices $M_2(k_{\mathfrak{p}})$ and a unique division algebra corresponding to the Hilbert symbol $(\frac{u,\pi}{k_{\mathfrak{p}}})$, where $u \in R_{\mathfrak{p}}^*$ is a unit and π is the uniformiser of $v_{\mathfrak{p}}$.*

Let now k be a number field and let $A = (\frac{a,b}{k})$ be a quaternion algebra over k. For any Galois element $\sigma \in \mathrm{Gal}(\mathbb{C})$ one can define the quaternion $\sigma(k)$-algebra $A^\sigma = (\frac{\sigma(a),\sigma(b)}{\sigma(k)})$.

If L/k is a field extension and $A = (\frac{a,b}{k})$, we can define the quaternion $L-$algebra $A \otimes_k L = (\frac{a,b}{L})$. In particular, for any valuation v on k we can define the local quaternion algebra $A_v = A \otimes_k k_v$, where k_v is the localisation of k with respect to v.

If A_v is isomorphic to $M_2(k_v)$ we say that A splits at the valuation v; otherwise we say that A ramifies at v. The subset of the set of places $\Omega(k)$ (resp. of the set of non-Archimedean places $\Omega_f(k)$, resp. of the set of Archimedean places $\Omega_\infty(k)$) consisting of the valuations at which A ramifies is denoted by $\mathrm{Ram}(A)$ (resp. $\mathrm{Ram}_f(A)$, resp. $\mathrm{Ram}_\infty(A)$). The valuations $v \in \mathrm{Ram}_f(A)$ will then correspond to certain primes ideals $\mathfrak{P}$, and we can define the *discriminant* of A as $D(A) = \prod_{v \in \mathrm{Ram}_f(A)} \mathfrak{P}$.

One can characterize quaternion algebras over number fields by looking at the places at which they ramify.

THEOREM 2.2. *Let A, A' be quaternion algebras over a number field k. They are isomorphic if and only if* $\mathrm{Ram}(A) = \mathrm{Ram}(A')$.

One can prove that $A \cong M_2(k)$ if and only if A_v splits for every $v \in \Omega(k)$. This is a consequence of the Hasse–Minkowski Theorem, which is a powerful local-global result on quadratic forms.

2.4. Algebras over p-adic fields. The tree of maximal orders

As before, given an order $\mathcal{O}$ in A, for each valuation v on k we can define the order $\mathcal{O}_v = \mathcal{O} \otimes_{R_k} R_v$, where R_v is the ring of integers of the local field k_v. One has the following global-local result (see for example [**17**], Lemma 6.2.7).

LEMMA 2.2. *Fix an order $\mathcal{I}$ in A. Given any other order $\mathcal{O}$ in A, for almost every non-Archimedean valuation v one has $\mathcal{O}_v = \mathcal{I}_v$. Moreover, there is a bijection*

$$\begin{array}{ccll} \{\textit{orders } \mathcal{O} \subset A\} & \longrightarrow & \{(\mathcal{L}_v)_{v \in \Omega_f(k)} : & \mathcal{L}_v \textit{ is an order in } A_v, \\ & & & \mathcal{L}_v = \mathcal{I}_v \textit{ for almost all } v \in \Omega_f(k)\} \\ \mathcal{O} & \longmapsto & (\mathcal{O}_v)_{v \in \Omega_f(k)} & \end{array}$$

This bijection preserves inclusion.

The fact that $\mathcal{O}_v$ and $\mathcal{I}_v$ coincide at almost all non-Archimedean localisations follows from Lemma 2.1, since the element $a \in R_k$ given by this lemma will be a unit for almost all $v_\mathfrak{p}$. It is also easy to prove that an order $\mathcal{O}$ is maximal if and only if all its non-Archimedean localisations are. Similarly it can be seen that being an Eichler order is also a local-global property.

As a consequence, most of the properties of maximal orders of quaternion algebras over number fields can be studied by looking at their non-Archimedean localisations. Conveniently, the p-adic situation is simpler.

Let $k_\mathfrak{p}$ be the localisation of the number field k at a finite place $\mathfrak{p}$. As we have seen the ring of integers $R_\mathfrak{p}$ has only one maximal ideal $\mathcal{P}$ generated by the uniformiser π of $v_\mathfrak{p}$.

Let A be a quaternion algebra over $k_\mathfrak{p}$. If A is the unique division algebra over $k_\mathfrak{p}$, then it has only one maximal order. On the other hand, if A is isomorphic to $M_2(k_p)$, then its maximal orders can be represented as vertices of a regular tree of valency $q+1$, where the norm q denotes the number of elements of the residue class field $R_\mathfrak{p}/\mathcal{P}$ (see [**27**] pp. 40–41). Two vertices are joined by an edge if and only if the corresponding maximal orders are conjugate by an element whose norm is in $R_p^*\mathcal{P}$ (see Figure 2.1).

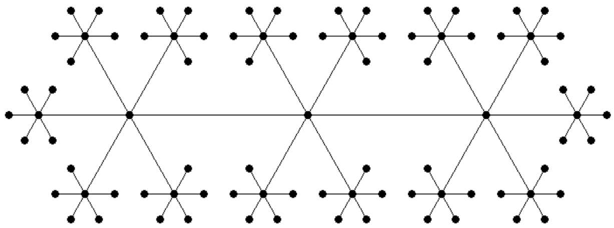


FIGURE 2.1. Part of the tree of local maximal orders for $q = 5$.

CHAPTER 3

Arithmetic Fuchsian groups

3.1. Theorems of Borel–Harish-Chandra and Margulis. Arithmetic triangle groups

Let Γ_1, Γ_2 be Fuchsian groups. They are said to be *commensurable* if their intersection has finite index in both of them, i.e. $[\Gamma_1 : \Gamma_1 \cap \Gamma_2] < \infty$ and $[\Gamma_2 : \Gamma_1 \cap \Gamma_2] < \infty$. The *commensurator group* of a Fuchsian group Γ is then defined as

$$\mathrm{Comm}(\Gamma) = \{\gamma \in \mathrm{PSL}(2, \mathbb{R}) : [\Gamma : \Gamma \cap \gamma\Gamma\gamma^{-1}] < \infty \text{ and } [\gamma\Gamma\gamma^{-1} : \Gamma \cap \gamma\Gamma\gamma^{-1}] < \infty\}.$$

We define the *invariant trace field* of Γ as the field $k_\Gamma = \mathbb{Q}(\mathrm{tr}(\Gamma^2))$ generated by the traces of the squares of elements of Γ. This is an invariant of the commensurability class of Γ, in other words any other Fuchsian group commensurable with Γ has the same invariant trace field.

The following is a consequence of a more general theorem by Borel and Harish-Chandra. Let k be a totally real number field, i.e. a number field all of whose embeddings $\sigma(k) \subset \mathbb{C}$ lie in $\mathbb{R}$. Let A be a quaternion algebra over k ramified at all infinite valuations but one, that is such that

$$\begin{array}{ll} A \otimes_k \mathbb{R} \cong M_2(\mathbb{R}) & \text{and} \\ A^\sigma \otimes_{\sigma(k)} \mathbb{R} \cong \mathcal{H} & \text{for every } \sigma \in \mathrm{Gal}(\mathbb{C}) \text{ with } \sigma \neq \mathrm{id}. \end{array}$$

Note that under these conditions, the injection ρ in equation (2.1) allows us to regard A as a subalgebra of $M_2(\mathbb{R})$. Let $\mathcal{O}$ be an order in A and write $\mathcal{O}^1$ for its norm 1 group. Then the subgroup $\mathrm{P}\rho(\mathcal{O}^1) \subset \mathrm{PSL}(2, \mathbb{R})$, where the P stands for the usual projection $\mathrm{SL}(2, \mathbb{R}) \to \mathrm{PSL}(2, \mathbb{R})$, is a Fuchsian group.

A Fuchsian group Γ is said to be an *arithmetic Fuchsian group* if it is commensurable with any such $\mathrm{P}\rho(\mathcal{O}^1)$.

The most classical example of an arithmetic Fuchsian group is $\mathrm{PSL}(2, \mathbb{Z})$. It is (the projective image of) the norm 1 group of the ring of matrices $M_2(\mathbb{Z})$, which is an order in the quaternion $\mathbb{Q}-$algebra $M_2(\mathbb{Q})$. Note that this quaternion algebra trivially satisfies the hypothesis above.

Very few among all Fuchsian groups are arithmetic, but they play a central role in many situations. One of the points in which they differ from the non-arithmetic Fuchsian groups is the following ([**18**]).

THEOREM (Margulis). *Let* Γ *be a Fuchsian group. Then* Γ *is non-arithmetic if and only if* $\mathrm{Comm}(\Gamma)$ *is an extension of finite index of* Γ. *Otherwise* $\mathrm{Comm}(\Gamma)$ *is dense in* $\mathrm{PSL}(2, \mathbb{R})$.

In the case when Γ is an arithmetic Fuchsian group, then the commensurator $\mathrm{Comm}(\Gamma)$ coincides with $\mathrm{P}\rho(\mathrm{A}^1)$.

Though in general it is difficult to know at first glance whether a given Fuchsian group is arithmetic or not, in the case of triangle groups the situation is completely known. In the 70's Takeuchi proved that there is only a finite number of arithmetic triangle groups and gave in [**26**] an exhaustive list of all such groups (both cocompact and non-cocompact), together with the inclusions between them.

3.2. Multiple dessins on an arithmetic Riemann surface

Let us recall our main objective. Given a uniform dessin corresponding to the inclusion $K < \Delta(l, m, n)$ of a torsion-free group K in an arithmetic triangle group $\Delta(l, m, n)$, we want to know if there exists $\beta \in \mathrm{PSL}(2, \mathbb{R})$ (and how many such elements are there, modulo K) such that

$$\begin{array}{ccc} \Delta(l,m,n) & & \beta^{-1}\Delta(l,m,n)\beta \\ & \diagdown \quad \diagup & \\ & K & \end{array}$$

Let us suppose for now that $\Delta(l,m,n)$ corresponds to the norm-1 group of a maximal order $\mathcal{M}$ in a quaternion algebra A, that is $\Delta(l,m,n) = \mathrm{P}\rho(\mathcal{M}^1)$. For simplicity we will often write $\mathcal{M}^1$ for $\Delta(l,m,n)$.

We want to study now common finite index subgroups K of $\mathcal{M}^1$ and $\beta^{-1}\mathcal{M}^1\beta$ and the possible conjugators β in this configuration. Clearly, conjugation by such a β induces an automorphism of the quaternion algebra, therefore the Skolem–Noether theorem allows to replace β with a more convenient element $\alpha \in A$. By multiplication with a denominator in the integers of k we can even suppose α to be in the maximal order $\mathcal{M}$.

Theorem 3.1. *Let $\mathcal{M}^1$ be the norm 1 group of a maximal order $\mathcal{M}$ and suppose that $\beta \in \mathrm{SL}(2,\mathbb{R})$ is such that $\mathcal{M}^1 \cap \beta^{-1}\mathcal{M}^1\beta$ has finite index in $\mathcal{M}^1$ and $\beta^{-1}\mathcal{M}^1\beta$. Then β can be replaced with a scalar multiple $\alpha \in \mathrm{GL}(2,\mathbb{R})^+ \cap \mathcal{M} \subset A$.*

Under these conditions $\mathcal{M}^1 \cap \alpha^{-1}\mathcal{M}^1\alpha$ is the norm 1 group of an Eichler order $\mathcal{E} = \mathcal{M} \cap \alpha^{-1}\mathcal{M}\alpha$ and one has

$$\begin{array}{ccc} \mathcal{M}^1 & & \alpha^{-1}\mathcal{M}^1\alpha \\ & \diagdown \quad \diagup & \\ & \mathcal{E}^1 & \\ & | & \\ & K & \end{array}$$

The index of $\mathcal{E}^1 = \mathcal{M}^1 \cap \alpha^{-1}\mathcal{M}^1\alpha$ in $\mathcal{M}^1$ gives a lower bound for $[\mathcal{M}^1 : K]$ where K denotes a surface group contained in both $\mathcal{M}^1$ and $\alpha^{-1}\mathcal{M}^1\alpha$.

For arithmetic triangle groups one has the additional advantage that all quaternion algebras in question have type number 1 ([**26**], Prop. 3), therefore all maximal orders are conjugate in A and all Eichler orders are intersections of conjugate maximal orders. So counting multiple dessins on $\mathbb{H}/K$ amounts to counting maximal orders containing $\hat{K}$, the preimage of K in $\mathrm{SL}(2,\mathbb{R})$.

3.3. Local-global arguments. Congruence subgroups

Maximal orders are easier to classify locally, i.e. over local fields, and the type number 1 property implies that there are bijections between

- prime ideals $\mathfrak{p}$ in the ring of integers R_k of the center k of the quaternion algebra A
- inequivalent primes elements π in R_k generating these prime ideals (without loss of generality we will suppose $\pi > 0$)
- inequivalent discrete valuations v of A
- inequivalent completions $A_v = A_\mathfrak{p}$ and $\mathcal{M}_v = \mathcal{M}_\mathfrak{p}$ of the quaternion algebra and a maximal order with respect to v

Recall that for a non-Archimedean valuation $v = v_\mathfrak{p}$, the local algebra A_v is a division algebra if and only if $\mathfrak{p}$ ramifies in A, i.e. if it belongs to the finite number divisors of the discriminant $D(A)$. In this case, $\mathcal{M}_\mathfrak{p}$ is the unique maximal order of $A_\mathfrak{p}$, and therefore there are no Eichler orders at all.

In all other (unramified) cases we get matrix algebras $A_\mathfrak{p} \cong M_2(k_\mathfrak{p})$, with maximal order $\mathcal{M}_\mathfrak{p} \cong M_2(R_\mathfrak{p})$ where $R_\mathfrak{p}$ denotes the ring of integers in the local field $k_\mathfrak{p}$, i.e. the completion of R_k in $k_\mathfrak{p}$. This ring has the unique prime ideal $\mathcal{P} = \pi R_\mathfrak{p}$, and all Eichler orders are conjugate to a ring of matrices

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{with} \quad a,b,d \in R_\mathfrak{p} \ , \ c \in \mathcal{P}^n \right\}$$

for some positive integer n ($\mathcal{P}^n$ is the *level* of the Eichler order). This local Eichler order is in fact an intersection $\mathcal{M}_\mathfrak{p} \cap \alpha^{-1}\mathcal{M}_\mathfrak{p}\alpha$ of two maximal orders conjugate by some $\alpha \in \mathcal{M}_\mathfrak{p}^*\left(\begin{smallmatrix} \pi & 0 \\ 0 & 1 \end{smallmatrix}\right) \subset M_2(R_\mathfrak{p})$.

The study of the local situation will be crucial to get global consequences using local-global arguments.

Suppose that the surface group K is included in both $\Delta = \mathcal{M}^1$ and $\alpha\Delta\alpha^{-1} = \alpha\mathcal{M}^1\alpha^{-1}$ for some $\alpha \in \mathrm{PSL}(2,\mathbb{R})$, so that K is included in the norm 1 group of the Eichler order $\mathcal{E} = \mathcal{M} \cap \alpha\mathcal{M}\alpha^{-1}$. The local situation is the following:

- For all valuations $v \in \mathrm{Ram}_f(A)$ the localised algebra A_v contains a unique maximal order, and therefore $\mathcal{E}_v = \mathcal{M}_v = \alpha\mathcal{M}_v\alpha^{-1}$;
- By Lemma 2.2 there is a finite number of $v \notin \mathrm{Ram}_f(A)$ such that $\mathcal{M}_v \neq \alpha\mathcal{M}_v\alpha^{-1}$.

Local Eichler orders in $M_2(k_v)$ are easy to study thanks to the tree structure of the maximal orders, mentioned in section 2.4. Recall that, if the valuation v corresponds to a prime ideal $\mathfrak{p}$, vertices in the tree correspond to maximal orders and two vertices are joined by an edge if and only if the corresponding maximal orders are conjugate under an element whose norm is in $R_k^*\mathfrak{p}$. The chain of inclusions

$$\mathcal{M}_v > \mathcal{M}_v \cap \left(\begin{smallmatrix}\mathfrak{p} & 0\\ 0 & 1\end{smallmatrix}\right)^{-1}\mathcal{M}_v\left(\begin{smallmatrix}\mathfrak{p} & 0\\ 0 & 1\end{smallmatrix}\right) > \ldots > \mathcal{M}_v \cap \left(\begin{smallmatrix}\mathfrak{p} & 0\\ 0 & 1\end{smallmatrix}\right)^{-n}\mathcal{M}_v\left(\begin{smallmatrix}\mathfrak{p} & 0\\ 0 & 1\end{smallmatrix}\right)^{n}$$

implies that a local Eichler order $\mathcal{E}_v = \mathcal{M}_v \cap \alpha\mathcal{M}_v\alpha^{-1}$ is contained in all the maximal orders corresponding to vertices lying in the path joining $\mathcal{M}_v$ and $\alpha\mathcal{M}_v\alpha^{-1}$. If $\mathcal{M}_v$ and $\alpha\mathcal{M}_v\alpha^{-1}$ are neighbours we will say that $\mathcal{E}$ is an Eichler order of level $\mathcal{P}$ and, more generally, if the path joining the two maximal orders has length n, we will say that $\mathcal{E}$ is an Eichler order of level $\mathcal{P}^n$.

We begin with the simplest case, which is local Eichler orders of level $\mathcal{P}$. Let q be the number of elements in the residue field $R_v/\mathcal{P} \cong \mathbb{F}_q$. We have the following result.

LEMMA 3.1. *Let $\mathcal{M}_v = M_2(R_v)$ be a local maximal order in $A_v = M_2(k_v)$. The norm 1 group $\Phi_0 = \Phi_0(\mathcal{P})$ of an Eichler order $\mathcal{M}_v \cap \alpha\mathcal{M}_v\alpha^{-1}$ of level $\mathcal{P}$ has index $q+1$ in $\mathcal{M}_v^1$. Moreover, $\mathcal{M}_v$ and $\alpha\mathcal{M}_v\alpha^{-1}$ are the only maximal orders in which Φ_0 is contained.*

PROOF. If one considers the canonical action of $\mathcal{M}_v^1 = M_2(R_v)$ on the projective line $\mathbb{P}^1(\mathbb{F}_q)$ given by reduction modulo $\mathcal{P}$, the groups Φ_0 correspond to the subgroups fixing one point. There are therefore $q+1$ of them, and this number coincides with the index. If the Eichler order was included in further maximal orders apart from $\mathcal{M}_v$ and $\alpha\mathcal{M}_v\alpha^{-1}$, it would correspond to a longer path in the tree of maximal orders, which is a contradiction since it has level $\mathcal{P}$. $\square$

Let $\mathcal{O} \subset A$ be a maximal order in the quaternion $k-$algebra A and $v \notin \mathrm{Ram}(A)$ an unramified valuation of A corresponding to the prime $\mathfrak{p}$, such that $\mathcal{O}_v = M_2(R_v)$. Let $\mathcal{E}(\mathfrak{p})$ denote the local Eichler order $\mathcal{M}_v \cap \left(\begin{smallmatrix}\mathfrak{p} & 0\\ 0 & 1\end{smallmatrix}\right)^{-1}\mathcal{M}_v\left(\begin{smallmatrix}\mathfrak{p} & 0\\ 0 & 1\end{smallmatrix}\right)$, whose norm 1 group we have denoted by $\Phi_0(\mathcal{P})$. We will write $\Delta_0(\mathfrak{p})$ for the norm 1 group of the Eichler order $\mathcal{E}$ in A that corresponds via the bijection in Lemma 2.2 to the family

$$\{(\mathcal{E}_v) : \mathcal{E}_v = \mathcal{E}(\mathfrak{p}) \text{ if } v = \mathfrak{p}, \text{ and } \mathcal{E}_v = \mathcal{O}_\mathfrak{p}, \text{ if } v \neq \mathfrak{p}\}.$$

In the particular case where $A = M_2(k)$ and R_k is a principal ideal domain all maximal orders are conjugate, and we can suppose that $\mathcal{O} = M_2(R_k)$. Therefore $\Delta_0(\mathfrak{p})$ coincides with the congruence subgroup

$$\Delta_0(\mathfrak{p}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Delta \subset M_2(R_k) : c \equiv 0 \bmod \mathfrak{p} \right\}.$$

The following lemma describes the norm 1 groups of the intersections of local Eichler orders of level $\mathcal{P}$.

LEMMA 3.2. *Let $\mathcal{M}_v = M_2(R_v)$ be a local maximal order in $A_v = M_2(k_v)$. Now we consider $\mathcal{M}_v^1$ and its subgroups as subgroups of $\mathrm{PSL}(2,R_v)$, i.e. modulo $\pm\mathrm{Id}$. Then*

(i) *The norm 1 group $\Phi_0^0 = \Phi_0^0(\mathcal{P})$ of the intersection of two Eichler orders of level $\mathcal{P}$ has index $q(q+1)$ in $\mathcal{M}_v^1$. Moreover, Φ_0^0 is contained in 3 different maximal orders if $q > 3$, 5 if $q = 3$ and 4 if $q = 2$.*

(ii) *The norm 1 group $\Phi(\mathcal{P})$ of the intersection of more than two Eichler orders of level $\mathcal{P}$ is the principal congruence subgroup modulo $\mathcal{P}$ of $\mathcal{M}_v^1$, a normal subgroup of $\mathcal{M}_v^1$ of index $\frac{1}{2}q(q^2-1)$ (omit the denominator 2 if q is a $2-$power). It is the intersection of all such Eichler orders of level $\mathcal{P}$ and is included in $q+2$ different maximal orders.*

PROOF. If we consider again the canonical operation of $\mathcal{M}_v^1$ on the projective line $\mathbb{P}^1(\mathbb{F}_q)$, the groups Φ_0^0 correspond to the elements fixing two points. If more than two points are fixed, automatically all points of the projective line are fixed, hence the case in (*ii*) already gives the principal congruence subgroup.

The cases $q = 2$ and 3 play a special role because for them $\Phi_0^0(\mathcal{P}) = \Phi(\mathcal{P})$: recall that we see them as projective groups, and since the determinants are 1, in the case of small q all matrices in $\Phi_0^0(\mathcal{P})$ are congruent $\mod \mathcal{P}$ to $\pm$ the unit matrix.

For the calculation of the indices one may consult [**27**] p. 109 or mimic a proof from any book about modular forms. Alternatively one may consider the groups involved as the stabilizers of one point, two points or the whole projective line, and then the index is given by the number of elements in the orbit of the fixed points. □

In a similar way to the case of $\Delta_0(\mathfrak{p})$ and $\Phi_0(\mathcal{P})$, we can define the principal congruence subgroup $\Delta(\mathfrak{p})$ as the subgroup of Δ whose localisation in $\mathfrak{p}$ coincides with $\Phi(\mathcal{P})$. The existence and uniqueness of such a subgroup is granted by the Strong Approximation Theorem (see for example [**27**] or [**17**]), which is an extreme version of the Chinese remainder theorem for certain matrix groups and whose formal statement exceeds the purposes of this course. In the particular case where $A = M_2(k)$ we can again suppose that $\mathcal{O} = M_2(R_k)$ and therefore

$$\Delta(\mathfrak{p}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod \mathfrak{p} \right\}.$$

LEMMA 3.3. *For integers $n > 1$ there are $q^{n-1}(q+1)$ different local Eichler orders $\mathcal{M}_v \cap \alpha^{-1}\mathcal{M}_v\alpha$ of level $\mathcal{P}^n$. Their norm 1 groups $\Phi_0(\mathcal{P}^n)$ have index $q^{n-1}(q+1)$ in $\mathcal{M}_v^1$. The intersection of all these norm 1 groups is the principal congruence subgroup $\Phi(\mathcal{P}^n)$, which is included in $\frac{(q+1)(q^n-1)}{q-1} + 1$ different maximal orders.*

PROOF. To prove that there are precisely $q^{n-1}(q+1)$ such Eichler orders of level $\mathcal{P}^n$ with norm 1 group $\Phi_0(\mathcal{P}^n)$ one may just count paths of length n in the tree of maximal orders, with one end fixed in the vertex $\mathcal{M}_v$. For the index formula one may use the same argument of the previous Lemma, this time defining an action of $\mathcal{M}_v^1$ on the "fake projective line" $\mathbb{P}_m^1$ over the residue class ring $R_k/\mathfrak{p}^m \cong R_v/\mathcal{P}^m$, which is the set of pairs of residue classes, not both in $\mathfrak{p}R_k/\mathfrak{p}^m$, modulo the unit group of this residue class ring (see also [**27**] p. 55).

The intersection of all Eichler orders of level $\mathcal{P}^n$ is then included in all the maximal orders at distance n from $\mathcal{M}_v$. □

As an illustration for the result concerning the principal congruence subgroups, we show in Figure 3.1 the picture of the subtree for $\Phi(\mathfrak{p}^2)$ in the case $q = 7$. We can define global principal congruence subgroups $\Delta(\mathfrak{p}^n)$ of higher level in the same way as above.

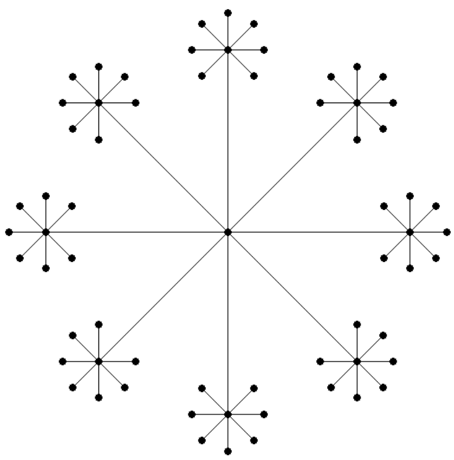


FIGURE 3.1. Subtree for $\Phi(\mathfrak{p}^2)$ in the local algebra $A_{\mathfrak{p}}$ for $q = 7$

3.4. Global consequences

We have the following necessary condition for the existence of at least two different uniform dessins of the same type on a Riemann surface of genus $g > 1$. This result will be crucial for the construction of low genus examples.

THEOREM 3.2. *Let K be an arithmetic Fuchsian surface group contained in the triangle group $\Delta = \Delta(l,m,n)$, and suppose that Δ is the norm 1 group $\mathcal{M}^1$ in a maximal order $\mathcal{M}$ of a quaternion algebra A defined over the totally real field k with ring of integers R_k. The group K is contained in more than one triangle group of type (l,m,n) if and only if K is contained in a group conjugate in Δ to $\Delta_0(\mathfrak{p})$, where $\mathfrak{p}$ is a prime of k not dividing the discriminant of A.*

PROOF. Suppose first that $K < \Delta \cap \alpha\Delta\alpha^{-1}$ for some $\alpha \in \mathrm{PSL}(2,\mathbb{R})$. We can suppose $\alpha \in A$ by the Skolem–Noether Theorem, and then by Lemma 2.2 there exists at least one valuation $\mathfrak{p} \notin \mathrm{Ram}(A)$ such that $\mathcal{M}_\mathfrak{p} \neq \alpha\mathcal{M}_\mathfrak{p}\alpha^{-1}$. For this valuation, we can suppose modulo conjugation that $\mathcal{M}_\mathfrak{p} \cap \alpha\mathcal{M}_\mathfrak{p}\alpha^{-1} \subset \mathcal{E}(\mathfrak{p})$, and therefore $K < \Delta \cap \alpha\Delta\alpha^{-1} < \Delta_0(\mathfrak{p})$.

The converse follows directly from the definitions. □

To understand Theorem 3.2 and construct examples in low genus, suppose that $K < \Delta_0(\mathfrak{p})$ for some $\mathfrak{p} \notin \mathrm{Ram}(A)$. This group $\Delta_0(\mathfrak{p})$ is always contained in the so called *Fricke extension* $\Delta_0^{\mathrm{Fr}}(\mathfrak{p})$ which, in the case of $A = M_2(k)$, is the index two extension of $\Delta_0(\mathfrak{p})$ by the element

$$\alpha = \frac{1}{\sqrt{\mathfrak{p}}}\begin{pmatrix} 0 & \mathfrak{p} \\ -1 & 0 \end{pmatrix} \in \mathrm{PSL}(2,\mathbb{R}),$$

where $\mathfrak{p}$ is chosen to be totally positive. This element clearly normalises $\Delta_0(\mathfrak{p})$, but not Δ. The action induced by conjugation on $\Delta_0(\mathfrak{p})$ is called the *Fricke involution.* As a consequence the group $K < \Delta_0(\mathfrak{p})$ is included in both Δ and $\alpha\Delta\alpha^{-1}$, yielding two different uniform dessins in $\mathbb{H}/K$. In the ramified case, the Fricke involution can be seen in the localised algebra $A_\mathfrak{p}$ as the element $\left(\begin{smallmatrix} 0 & 1/\mathfrak{p} \\ -1 & 0 \end{smallmatrix}\right)$, which interchanges by conjugation $\mathcal{M}_v$ and $\left(\begin{smallmatrix} \mathfrak{p} & 0 \\ 0 & 1 \end{smallmatrix}\right)^{-1}\mathcal{M}_v\left(\begin{smallmatrix} \mathfrak{p} & 0 \\ 0 & 1 \end{smallmatrix}\right)$, and therefore fixes $\mathcal{E}(\mathfrak{p})$.

Now we will concentrate on a series of striking examples. Take Δ of signature $(2,3,7)$. According to [**26**] this is the norm 1 group of a maximal order $\mathcal{M}$ in a quaternion algebra A over the cubic field $k = \mathbb{Q}(\cos\frac{2\pi}{7})$.

It is well known that Hurwitz curves are uniformised by normal subgroups K of the triangle group $\Delta(2,3,7)$ and that, in particular, one has $\mathrm{Aut}(S) \cong \Delta(2,3,7)/K$. A classical theorem by Macbeath ([**15**]) shows that $\mathrm{PSL}(2,\mathbb{F}_q)$ is a Hurwitz group exactly in the following cases

(i) $q = 7$,
(ii) $q = p$ prime for $p \equiv \pm 1 \bmod 7$,
(iii) $q = p^3$ for p prime and $p \equiv \pm 2$ or $\pm 3 \bmod 7$.

Accordingly, the corresponding Riemann surfaces are usually known as *Macbeath–Hurwitz curves.*

It was proved in [**4**] by A. Džambić that all Macbeath-Hurwitz curves can be constructed arithmetically as follows. The triangle group $\Delta(2,3,7)$ is the norm 1 group of a maximal order in the quaternion A over the field $k = \mathbb{Q}(\cos\pi/7)$ which is ramified exactly over the two non-trivial Archimedean valuations of k. Any rational prime p defines an ideal $p\mathcal{R}_k$ in $\mathcal{R}_k$ such that

(i) if $p = 7$ then p is ramified and $p\mathcal{R}_k = \mathfrak{p}^3$ for a prime ideal $\mathfrak{p} \subset \mathcal{R}_k$ of norm $q = N(\mathfrak{p}) = 7$;
(ii) if $p \equiv \pm 1 \bmod 7$ then p splits, i.e. $p\mathcal{R}_k = \mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_3$ for prime ideals $\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3 \subset R_k$ of norm $q = N(\mathfrak{p}_i) = p$;
(iii) if $p \equiv \pm 2$ or $\pm 3 \bmod 7$ then p is inert, i.e. $p\mathcal{R}_k$ is a prime ideal in $\mathcal{R}_k$ of norm $q = N(\mathfrak{p}) = p^3$.

For every prime $\mathfrak{p}$ in $\mathcal{R}_k$ we can define the principal congruence subgroup of $\Delta(2,3,7)$ corresponding to the prime $\mathfrak{p}$. This is a normal torsion-free subgroup of $\Delta(2,3,7)$ with quotient group isomorphic to $\mathrm{PSL}(2,\mathbb{F}_q)$ where $q = N(\mathfrak{p})$, yielding therefore a Macbeath-Hurwitz curve.

The first cases are:

- Klein's quartic. Its surface group is $\Delta(\mathfrak{p})$ for a prime $\mathfrak{p}$ dividing 7, ramified of order 3 and of residue degree 1 in the extension $\mathbb{Q}(\cos\frac{2\pi}{7})/\mathbb{Q}$. With $q = 7$ we see that Klein's quartic has 8 conjugate uniform dessins of type $(2,3,7)$ plus the usual regular one.

- Macbeath's curve of genus 7 with automorphism group $\mathrm{PSL}(2,\mathbb{F}_8)$ has the surface group $\Delta(2)$ for the prime $\mathfrak{p}=2$, inert and of residue degree 3 in the extension $\mathbb{Q}(\cos\frac{2\pi}{7})/\mathbb{Q}$. With $q=8$ one has 9 uniform dessins plus a regular one on the curve.
- Three non-isomorphic curves in genus 14 whose automorphism groups are isomorphic to $\mathrm{PSL}(2,\mathbb{F}_{13})$ and whose surface groups are the principal congruence subgroups $\Delta(\mathfrak{p}_j)$, $j=1,2,3$ for the (completely decomposed) primes $\mathfrak{p}_j$ dividing 13. Their residue degree is 1, hence one has $q+1=14$ uniform dessins of type $(2,3,7)$ on each curve plus a regular one.

All dessins mentioned here are clearly not renormalisations of each other since the signature consists of three different entries. On the other hand, in all these cases we have one regular dessin and $q+1$ uniform non-regular ones which form an orbit under the automorphism group of the curve: the $q+1$ norm 1 groups of type $\Delta_0(\mathfrak{p})$ are conjugate under the action of Δ (in other words, the $q+1$ Eichler orders of level $\mathcal{P}$ form a $\Delta-$invariant set), so these dessins are equivalent under automorphisms of the curve.

One can consider the growth of the maximal number of uniform dessins on surfaces $\mathbb{H}/K$, as a function of the index $[\Delta:K]$ in a given triangle group. Global-local arguments yield the following bound.

Theorem 3.3. *Let the Fuchsian group Δ be the norm 1 group of a maximal order in a quaternion algebra. For each positive integer $m>0$, the maximum number of conjugates of Δ in which any Fuchsian group $K<\Delta$ of index at most can be included is $O(\sqrt[3]{m})$ and this upper bound is optimal in the following sense. There are sequences of surface groups $K_n<\Delta$ with indices $[\Delta:K_n]\to\infty$ such that, if we write d_n for the number of all residue classes $\alpha\in\mathrm{PSL}(2,\mathbb{R})/N(\Delta)$ with the property $K_n\subset\alpha\Delta\alpha^{-1}$, we have*

$$\lim_{n\to\infty}\frac{d_n}{\sqrt[3]{2[\Delta:K_n]}}=1.$$

The proof of this result follows from considering local bounds and applying a local-global argument based on the Strong Approximation Theorem. For the sequence K_n one may take any sequence of principal congruence subgroups $\Delta(\mathfrak{p})$ with prime ideals $\mathcal{P}=\mathfrak{p}R_v$ such that $R_k/\mathfrak{p}\cong\mathbb{F}_q$, with $q\to\infty$. Observe that only finitely many among the K_n can have torsion.

However, in these examples we have only the rather modest number of two essentially different (non-isomorphic) dessins of the same type. Nevertheless, replacing these congruence groups with subgroups of small index we can remove automorphisms such that most of the uniform dessins found here become inequivalent. As a consequence, and describing the growth result given in Theorem 3.3 in terms of the genus, we get the following corollary.

Corollary 3.1. *The number of uniform dessins not equivalent under renormalisation or automorphisms on a Belyi surface grows with the genus g at most as a multiple of $\sqrt[3]{g}$, and this bound is optimal.*

We refer to [**9**] for full details of the proofs.

3.5. Examples in low genus

We explore now the examples given in section 3.4 in a more geometrical way.

3.5.1. Klein's quartic. Klein's quartic is a genus three surface uniformised by a group K generated by certain side-pairings in the regular 14-gon P with angle $2\pi/7$ (see Figure 3.2). The (black and white) triangles in Klein's original picture are related to the triangle group $\Delta(2,3,7)$ of signature $(2,3,7)$ in which K is normally contained with index 168.

The inclusion $K\lhd\Delta(2,3,7)$ induces a regular Belyi function on K. The corresponding regular dessin $\mathcal{D}$ can be easily depicted in P with the help of the triangle tessellation associated to $\Delta(2,3,7)$ (see left picture on Figure 3.3).

Now rotate $\mathcal{D}$, or rather its lift to the universal covering $\mathbb{D}$, by an angle $2\pi/14$ around the origin. The graph $\mathcal{D}'$ obtained is compatible with the side-pairing identifications, hence it is a well defined dessin on the surface. It is rather obvious that $\mathcal{D}'$ decomposes the surface into 24 heptagons in the same way as $\mathcal{D}$ does. In other words $\mathcal{D}'$ is also a uniform $(2,3,7)$ dessin on $\mathbb{H}/K$ (see right picture on Figure 3.3). Note that the rotation that transforms $\mathcal{D}$ into $\mathcal{D}'$ does

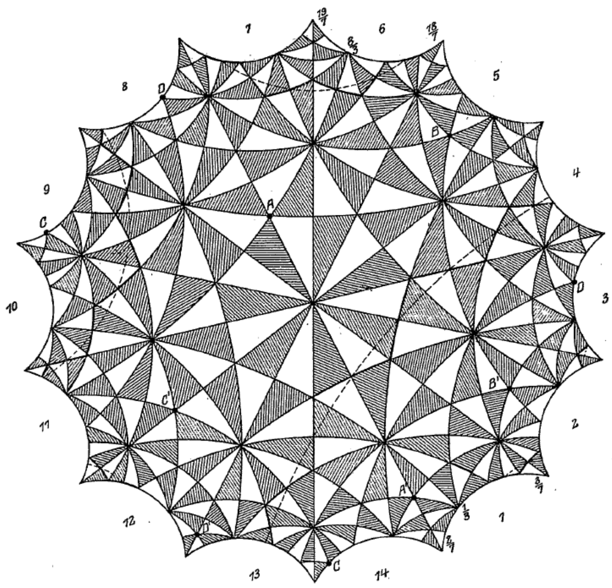


FIGURE 3.2. Klein's surface is obtained by the side pairing $1 \leftrightarrow 6$, $3 \leftrightarrow 8$, $5 \leftrightarrow 10$, $7 \leftrightarrow 12$, $9 \leftrightarrow 14$, $11 \leftrightarrow 2$, $13 \leftrightarrow 4$.

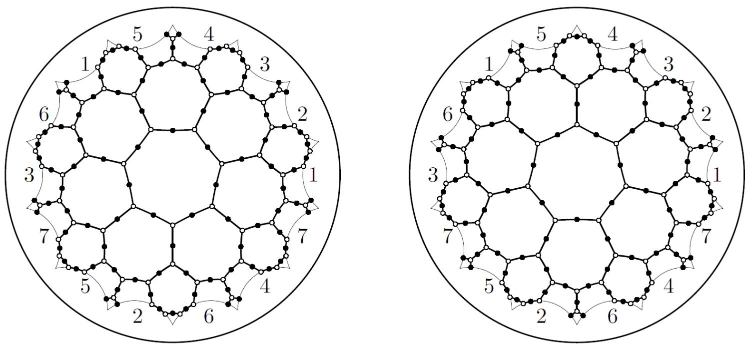
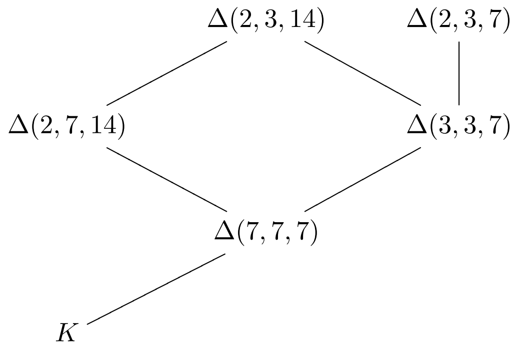


FIGURE 3.3. Klein's regular $(2,3,7)$ dessin $\mathcal{D}$ and a uniform one $\mathcal{D}'$.

not correspond to any automorphism of the surface, and in fact both dessins are not isomorphic since $\mathcal{D}'$ is not regular (it can be checked that the automorphism group $\mathrm{Aut}(S)$ does not act transitively on the edges of this new dessin).

This existence of a new uniform dessin of type $(2,3,7)$ is clear if one studies all triangle groups in which K is contained. The group K corresponds to $\Delta(\mathfrak{p})$, for a prime $\mathfrak{p}$ dividing 7 in $\mathbb{Q}(\cos\pi/7)$. The surface group K is a normal subgroup of $\Delta(2,3,7)$, but it is also contained normally in the group $\Delta(7,7,7)$ that has one seventh of the 14-gon as fundamental domain. The corresponding regular $(7,7,7)-$dessin lies in the border of the polygon: it has one black vertex, one white vertex, and seven edges. There is even a group $\Delta(3,3,7)$ lying between $\Delta(7,7,7)$ and $\Delta(2,3,7)$ that defines another regular dessin of type $(3,3,7)$. The chain of inclusions $K < \Delta(7,7,7) < \Delta(3,3,7) < \Delta(2,3,7)$ means that the corresponding regular dessins are related by refinement. Moreover, it can be checked that this group $\Delta(3,3,7)$ corresponds to $\Delta_0(\mathfrak{p})$, and therefore one has an index two extension $\Delta_0^{\mathrm{Fr}}(\mathfrak{p}) = \Delta(2,3,14)$. In fact, the full diagram of triangle groups lying above K can be found looking at Singerman's inclusion list:

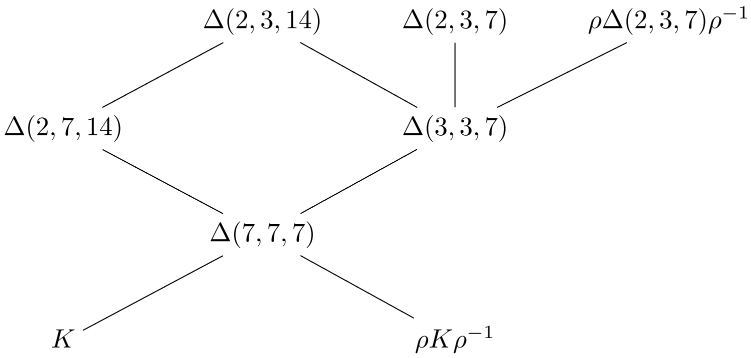
(3.1)



The groups $\Delta(2,7,14)$ and $\Delta(2,3,14)$ are the index two (therefore normal) extensions of $\Delta(7,7,7)$ and $\Delta(3,3,7)$ obtained by addition of a new element ρ (which induces the Fricke involution of $\Delta_0(\mathfrak{p})$) which is a rotation of angle $2\pi/14$ around the origin. The corresponding dessins of type $(2,7,14)$ and $(2,3,14)$ are not regular but only uniform (as already noticed in [**24**]), and are obtained from those of types $(7,7,7)$ and $(3,3,7)$ by colouring all the vertices with the same colour, say black, and then adding white vertices at the midpoints of the edges.

Conjugation of diagram (3.1) by ρ fixes all the groups except K and $\Delta(2,3,7)$:

(3.2)



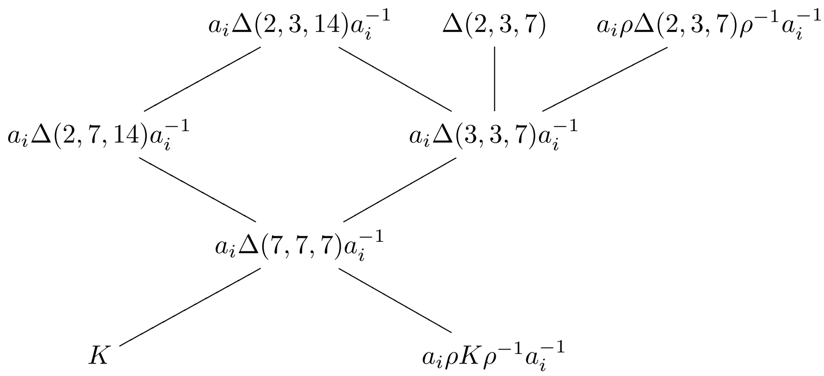
The inclusion $K < \rho\Delta(2,3,7)\rho^{-1}$ corresponds to the uniform dessin $\mathcal{D}'$ described above. Since the normaliser of K is $\Delta(2,3,7)$ the inclusion of K in the triangle group $\rho\Delta(2,3,7)\rho^{-1}$ is not normal, hence $\mathcal{D}'$ is not regular.

Now we focus in the group $\Delta(3,3,7)$ lying in the middle of diagrams (3.1) and (3.2). It is a known fact ([**10**]) that a given triangle group of type $(3,3,7)$ is contained in precisely two different groups of signature $(2,3,7)$, i.e. $\Delta(2,3,7)$ and $\rho\Delta(2,3,7)\rho^{-1}$ in our case. Conversely, any given $\Delta(2,3,7)$ contains eight different subgroups of signature $(3,3,7)$, all conjugate in $\Delta(2,3,7)$. From the point of view of local quaternion algebras, this is a consequence of Lemmas 3.1 and 3.2.

Let $a_0\Delta(3,3,7)a_0^{-1} = \Delta(3,3,7), a_1\Delta(3,3,7)a_1^{-1}, \ldots, a_7\Delta(3,3,7)a_7^{-1}$ be the 8 subgroups of $\Delta(2,3,7)$ conjugate to $\Delta(3,3,7)$, with $a_i \in \Delta(2,3,7)$.

If we conjugate diagram (3.2) by a_i we get

(3.3)



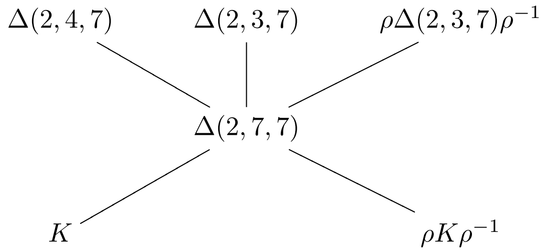
Note that only $\Delta(2,3,7)$ and K remain fixed by this conjugation, since a_i belongs to $\Delta(2,3,7)$, the normaliser of K.

The inclusion $K < a_i\rho\Delta(2,3,7)\rho^{-1}a_i^{-1}$ induces a new uniform (but not regular) dessin of type $(2,3,7)$ on $\mathbb{H}/K$. It is related to the uniform dessin $\mathcal{D}'$ by the automorphism induced by a_i, and to the regular dessin $\mathcal{D}$ by a hyperbolic rotation of angle $2\pi/14$ around the center of certain face of $\mathcal{D}$.

3.5.2. Macbeath's curve of genus seven. The description of the uniform $(2,3,7)$ dessins on Macbeath curve goes more or less along the same lines as in the case of Klein's quartic. Again the surface group K is included normally in $\Delta(2,3,7)$. The role played by the group $\Delta(3,3,7)$ in Klein's quartic is played here by $\Delta(2,7,7)$, which this time corresponds to $\Delta_0(2)$. Note that the inclusion $\Delta(2,7,7) < \Delta(2,3,7)$ is also very special (cf. [**10**]). The number of conjugate subgroups of type $(2,7,7)$ inside $\Delta(2,3,7)$ is nine, and any given $\Delta(2,7,7)$ is contained in two different groups of type $(2,3,7)$ (this is again consequence of Lemmas 3.1 and 3.2. The normaliser of $\Delta(2,7,7)$ is now a $(2,4,7)$–group obtained by adding a rotation ρ of order 4 around any of the points of order 2 in $\Delta(2,7,7)$.

This new element does not normalise $\Delta(2,3,7)$, so conjugation by ρ gives rise to the second group $\rho\Delta(2,3,7)\rho^{-1}$ in which $\Delta(2,7,7)$ is included:

(3.4)



The inclusion of K inside $\Delta(2,3,7)$ and $\rho\Delta(2,3,7)\rho^{-1}$ determines two non isomorphic dessins on Macbeath's curve. Once more the second inclusion is not normal, and accordingly the second dessin is uniform but not regular.

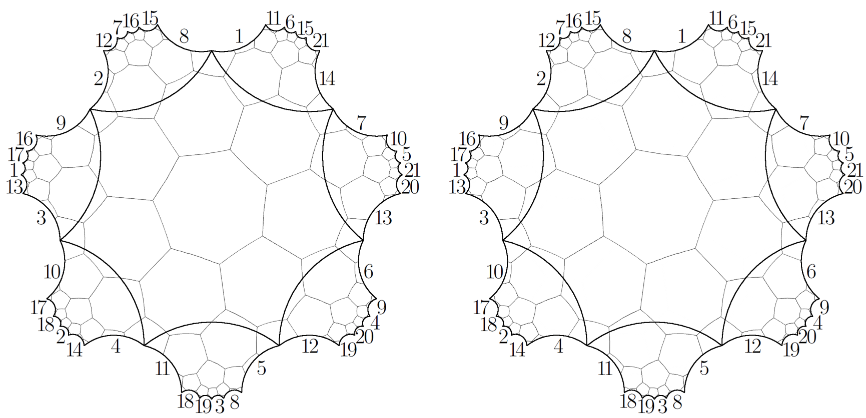


FIGURE 3.4. Face decomposition associated to regular and uniform dessins of type $(2,3,7)$ on Macbeath's surface.

We can proceed in the same way with the other eight $(2,7,7)$–groups contained inside $\Delta(2,3,7)$ to get diagrams similar to diagram (3.3). This way we find the nine (isomorphic) uniform dessins predicted by the arithmetic arguments of Section 3.4. There is obviously as well a uniform dessin of type $(2,4,7)$, as already noticed in [**24**].

3.5.3. Macbeath-Hurwitz curves of genus 14. The third example given in section 3.4 arises from the consideration of the three (torsion free) groups $K_i = \Delta(\mathfrak{p}_i) \lhd \Delta(2,3,7)$ for inequivalent primes $\mathfrak{p}_1$, $\mathfrak{p}_2$ and $\mathfrak{p}_3$ dividing 13 in $\mathbb{Q}(\cos\frac{\pi}{7})$. These groups correspond to three Galois conjugate curves of genus 14 with a regular $(2,3,7)$ dessin ([**25**]).

Now for each of these primes, we find $\Delta_0(\mathfrak{p}_i)$ lying between $\Delta(\mathfrak{p}_i)$ and $\Delta(2,3,7)$. Its index inside $\Delta(2,3,7)$ is 14. By Singerman's method for the determination of signatures of subgroups of Fuchsian groups ([**22**]) it can be seen that $\Delta_0(\mathfrak{p}_i)$ is a group of signature $\langle 0; 2, 2, 3, 3\rangle$.

There is again an element ρ_i in the normaliser of $\Delta_0(\mathfrak{p}_i)$ that conjugates $\Delta(2,3,7)$ into a different group. The inclusion of $\Delta(\mathfrak{p}_i)$ inside $\rho_i\Delta(2,3,7)\rho_i^{-1}$ is no longer normal and gives rise to a non-regular uniform dessin on the same Riemann surface.

Moreover, $\Delta(2,3,7)$ contains 14 different subgroups conjugate to $\Delta_0(\mathfrak{p}_i)$. All of them include $\Delta(\mathfrak{p}_i)$, therefore arguing as above we find 14 isomorphic uniform $(2,3,7)$ dessins.

Bibliography

[1] Beardon, A. F.: The geometry of discrete groups, Springer-Verlag, New York (1983);

[2] Belyĭ, G. V. : On Galois extensions of a maximal cyclotomic field, Izv. Akad. Nauk SSSR Ser. Mat. **43** (1979) 269–276 (in Russian); Math. USSR Izv. **14** (1980) 247–256 (in English).

[3] Cohen, P. B., Itzykson, C., Wolfart, J.: Fuchsian triangle groups and Grothendieck dessins. Variations on a theme of Belyi., Comm. Math. Phys. **163**, no. 3 (1994), 605–627.

[4] Džambić, A.: Macbeath's infinite series of Hurwitz groups. In: Holzapfel, R.–P., Uludag, A.M.; Yoshida, M. (eds.) Arithmetic and geometry around hypergeometric functions, pp. 101–108. Progr. Math., 260, Birkhäuser, Basel (2007).

[5] Earle, C. J.: On the moduli of closed Riemann surfaces with symmetries. Advances in the theory of riemann surfaces (Proc. Conf., Stony Brook, N.Y., 1969), pp. 119–130. Ann. of Math. Studies, No. 66, Princeton Univ. Press, Princeton, N.J., 1971.

[6] Farkas, H. M., Kra, I.: Riemann surfaces. Graduate Texts in Mathematics **71**, Springer-Verlag, New York-Berlin, 1980.

[7] Girondo, E.: Multiply quasiplatonic Riemann surfaces, Exp. Math. **12** (4) (2003), 463–475.

[8] Girondo, E., González-Diez, G.: Introduction to compact Riemann surfaces and dessins d'enfants, London Mathematical Society Student Texts **79**, Cambridge University Press, Cambridge, 2011.

[9] Girondo, E., Torres-Teigell, D., Wolfart, J.: Shimura curves with many uniform dessins, Math. Z., DOI: 10.1007/s00209-011-0889-4 (2011).

[10] Girondo, E., Wolfart, J.: Conjugators of Fuchsian Groups and Quasiplatonic Surfaces, Quart. J. Math. **56** (2005), 525–540.

[11] González-Diez, G.: Variations on Belyi's theorem, Quart. J. Math. **57** (2006), 355–366.

[12] Grothendieck, A.: Esquisse d'un Programme. In: Lochak, P., Schneps, L. (eds.) Geometric Galois Actions 1. Around Grothendieck's Esquisse d'un Programme, pp. 5–84, London Math. Soc. Lecture Note Ser. 242, Cambridge University Press (1997).

[13] Jones, G. A., Singerman, D.: Complex functions: an algebraic and geometric viewpoint, Cambridge University Press (1997);

[14] Katok, S.: Fuchsian groups. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1992.

[15] Macbeath, A. M.: Generators of the linear fractional groups. 1969 Number Theory (Proc. Sympos. Pure Math., Vol. XII, Houston, Tex., 1967) Amer. Math. Soc., Providence, R.I., 1969, pp. 14–32.

[16] Maclachlan, C.: Introduction to arithmetic Fuchsian groups. Topics on Riemann surfaces and Fuchsian groups, 29–41, London Math. Soc. Lecture Note Ser., 287, Cambridge Univ. Press, Cambridge, 2001.

[17] Maclachlan, C., Reid, A. W.: The arithmetic of hyperbolic 3-manifolds. Graduate Texts in Mathematics, **219**. Springer-Verlag, New York, 2003.

[18] Margulis, G.: Discrete subgroups of semisimple Lie groups. Springer–Verlag (1991).

[19] Miranda, R.: Algebraic curves and Riemann surfaces. Graduate Studies in Mathematics **5**, American Mathematical Society, Providence, 1995.

[20] Schneps, L. (editor): The Grothendieck theory of dessins d'enfants. London Mathematical Society Lecture Note Series, **200**. Cambridge University Press, Cambridge, 1994.

[21] Shimura, G.: On the field of rationality for an abelian variety, Nagoya Math. J. **45** (1972), 167–178.

[22] Singerman, D.: Subgroups of Fuchsian groups and finite permutation groups, Bull. London Math. Soc. **2** (1970), 319–323.

[23] Singerman, D.: Finitely maximal Fuchsian groups, J. London Math. Soc. (2) **6** (1972) 29–38.

[24] Singerman, D., Syddall, R.I.: The Riemann Surface of a Uniform Dessin, Beitr. zur Algebra und Geom. **44** (2003), 413–430.

[25] Streit, M.: Field of definition and Galois orbits for the Macbeath-Hurwitz curves, Arch. Math. (Basel) **74**, no. 5 (2000) 342–349.

[26] Takeuchi, K.: Commensurability classes of arithmetic triangle groups, J. Fac. Sci. Univ. Tokio, Sect. 1A Math. (1) **24** (1977), 201–212.

[27] Vignéras, M.–F.: Arithmétique des Algèbres de Quaternions, LNM 800, Springer (1980).

[28] Weil, A.: The field of definition of a variety. Amer. J. Math. 78 (1956), 509-524.

[29] Wolfart, J.: *ABC* for polynomials, dessins d'enfants and uniformization — a survey. Elementare und analytische Zahlentheorie, Schr. Wiss. Ges. Johann Wolfgang Goethe Univ. Frankfurt am Main, 20, Franz Steiner Verlag Stuttgart, Stuttgart, 2006, pp. 313–345.