UNIVERSIDAD AUTONOMA DE MADRID
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Triangle groups, dessins d'enfants and Beauville surfaces

Grupos triangulares, dessins d'enfants y superficies de Beauville

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Resumen y conclusiones

Una superficie de Riemann es una variedad de dimensión 2 en la que los cambios de cartas son funciones holomorfas entre abiertos del plano complejo. Las superficies de Riemann son siempre orientables, y por lo tanto las compactas están caracterizadas topológicamente por su género. Las superficies de Riemann compactas se pueden ver también como curvas algebraicas lisas sobre los complejos, y por lo tanto se puede definir una acción del grupo de Galois $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ sobre el conjunto de las superficies de Riemann compactas mediante la acción de los elementos de Galois en los polinomios con coeficientes complejos.

Otra forma de estudiar superficies de Riemann es desde el punto de vista de la uniformización. Por la teoría de espacios recubridores toda superficie de Riemann es el cociente de una superficie simplemente conexa, llamada recubridor universal, por la acción libre de un subgrupo del grupo de automorfismos de este recubridor. El teorema de uniformización nos asegura que toda superficie de Riemann simplemente conexa es isomorfa al plano, a la esfera o al disco unitario, y por lo tanto estos son los únicos posibles recubridores universales.

Si el género de una superficie compacta es mayor o igual que 2, el recubridor universal es necesariamente el disco, cuyo grupo de automorfismos es isomorfo a $\mathrm{PSL}(2,\mathbb{R})$. Una de las principales características de este grupo de automorfismos es que coincide con el grupo de isometrías del disco con la métrica hiperbólica, y por lo tanto cualquier superficie de Riemann de género mayor o igual que dos hereda de forma natural una métrica hiperbólica. Los subgrupos de $\mathrm{PSL}(2,\mathbb{R})$ que definen una superficie de Riemann en el cociente no tienen porqué actuar libremente, basta con que actúen de manera propiamente discontinua. A tales grupos se les llama grupos Fuchsianos.

Entre los grupos Fuchsianos, una familia importante es la de los grupos triangulares, que son grupos generados por giros alrededor de los tres vértices de un triángulo hiperbólico y que definen en el cociente una superficie de Riemann de género 0 con tres puntos marcados. Los grupos triangulares están estrechamente relacionados tanto con los dessins d'enfants como con las superficies de Beauville, que son los objetos principales de estudio de esta memoria.

Un dessin d'enfant es un grafo finito bicoloreado en una superficie topológica compacta y orientable cuyo complementario es unión finita de discos topológicos. Todo dessin dota a la superficie topológica en la que está inmerso de una estructura de superficie de Riemann. Es más, por el teorema de Belyi–Grothendieck esa superficie corresponde a una curva algebraica con coeficientes en el cuerpo de números algebraicos $\overline{\mathbb{Q}}$, y a la inversa, a toda curva con coeficientes algebraicos le corresponde al menos un dessin.

Por otro lado, una superficie de Beauville es una superficie compleja (variedad de dimensión real 4) isomorfa al cociente del producto de dos superficies de Riemann compactas por cierta acción libre de un grupo finito que actúa por isomorfismos. A esta acción se le pide, adicionalmente, que restrinja a cada una de las dos superficies de Riemann definiendo en el cociente una superficie de género 0 con tres puntos marcados. La relevancia de estas superficies viene principalmente del hecho de que son rígidas, es decir que no admiten deformaciones no triviales.

Aparte del capítulo de preliminares, en el que se exponen los conceptos necesarios para poder desarrollar los capítulos siguientes, el resto del texto se puede dividir en tres partes claramente diferenciadas.

En el capítulo 1 se definen los grupos triangulares, que están en la base de la mayoría de los conceptos que se introducen en los capítulos 2 y 3, y se describen sus propiedades. En particular demostramos con métodos elementales de geometría hiperbólica el conocido hecho que todo isomorfismo de grupos de un grupo triangular está inducido por una isometría del plano hiperbólico (Corolario 1.1). Estudiamos la relación entre grupos triangulares, dessins d'enfants y curvas triangulares (o cuasiplatónicas) y describimos la acción del grupo de Galois absoluto en $G-$cubrimientos triangulares desde el punto de vista de las tripletas que los definen (Proposición 1.3), formalizando un método de M. Streit. Como consecuencia demostramos el resultado ya conocido de que los $G-$cubrimientos triangulares abelianos están definidos sobre $\mathbb{Q}$ (Corolario 1.2). Finalmente, caracterizamos todos los cubrimientos triangulares con grupo $\mathrm{PSL}(2,p)$ y tipos (p,p,p) y $(2,3,n)$ (Teoremas 1.2 y 1.3) y los cubrimientos triangulares con grupo $\mathrm{PSL}(2,7)$ y tipo $(3,3,4)$ (Teorema 1.4).

En el capítulo 2 se estudia la existencia de múltiples dessins uniformes del mismo tipo en una superficie de Riemann. En el caso no aritmético el resultado es inmediato (Teorema 2.1). En el caso en el que el grupo que uniformiza la superficie es aritmético, mediante el estudio de órdenes maximales en álgebras de cuaterniones encontramos una condición necesaria y suficiente para que una superficie de Riemann contenga varios dessins uniformes (Teorema 2.3). También exponemos varios ejemplos de superficies de Riemann bien conocidas en las que, por los resultados anteriores, demostramos que viven varios dessins uniformes del mismo tipo (sección 2.4). Por último explicamos otro método para encontrar múltiples dessins uniformes en la misma superficie, que lleva a la caracterización de dessins uniformes unicelulares en género 2 (Teorema 2.5).

En el capítulo 3 se introduce el concepto de superficie de Beauville y se enumeran las propiedades de estas superficies complejas. En las primeras secciones se presenta la teoría de estas superficies desde el punto de vista de la uniformización, exponiendo resultados de Catanese desde otra perspectiva (secciones 3.1 y 3.2). En estas secciones también se incluyen resultados nuevos, como restricciones a los géneros que pueden tener las curvas que definen la superficie de Beauville (Proposición 3.1 y Teorema 3.1 en el caso no mixto, y Corolario 3.4 en el caso mixto) o consideraciones sobre el grupo de automorfismos de una superficie de Beauville (Teorema 3.2 en el caso no mixto y Teorema 3.3 en el caso mixto). En la sección 3.3 se demuestran resultados equivalentes a los teoremas de rigidez de superficies de Beauville de Catanese desde el punto de vista de grupos Fuchsianos (Teorema 3.4). Finalmente construimos varios ejemplos de superficies de Beauville no homeomorfas que son conjugadas Galois: primero el ejemplo con género mínimo y grupo de Beauville $\mathrm{PSL}(2,7)$ (Teorema 3.6 y Corolario 3.7) y después una familia infinita de superficies de Beauville con grupo de Beauville $\mathrm{PSL}(2,p)$ (Teoremas 3.7 y 3.8).

Los resultados principales de esta memoria pueden encontrarse en los siguientes artículos:

- E. Girondo, D. Torres-Teigell: *Genus 2 Belyĭ surfaces with a unicellular uniform dessin*, Geom. Dedicata **155** (2011), 81–103.
- E. Girondo, D. Torres-Teigell, J. Wolfart: *Shimura curves with many uniform dessins*, Math. Z. (2011), doi:10.1007/s00209-011-0889-4.
- G. González-Diez, D. Torres-Teigell: *An introduction to Beauville surfaces via uniformization*, en "Quasiconformal Mappings, Riemann Surfaces, and Teichmüller Spaces", Contemp. Math. (2012), pendiente de publicación.
- G. González-Diez, D. Torres-Teigell: *Non-homeomorphic Galois conjugate Beauville structures on* $\mathrm{PSL}(2,p)$, Adv. Math. (2012), doi:10.1016/j.aim.2012.02.014.

Los siguientes artículos también incluyen resultados relacionados con los contenidos de la memoria:

- G. González-Diez, G. A. Jones, D. Torres-Teigell: *Beauville surfaces with abelian Beauville group*, arXiv:1102.4552v1.
- G. González-Diez, G. A. Jones, D. Torres-Teigell: *Arbitrarily large Galois orbits of non-homeomorphic surfaces*, arXiv:1110.4930.

Summary and results

A Riemann surface is a manifold of dimension 2 for which the transition functions are holomorphic functions between open sets of the complex plane. Riemann surfaces are always orientable, and therefore the compact ones are topologically characterized by their genus. Compact Riemann surfaces can also be seen as smooth algebraic curves over the complex field, hence one can define an action of the Galois group $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ on the set of compact Riemann surfaces through the action of the Galois elements on the polynomials with complex coefficients.

One can also study Riemann surfaces from the point of view of uniformisation. By covering space theory every Riemann surface is the quotient of a simply connected surface, called the universal cover, by the free action of a subgroup of the group of automorphisms of this cover. The Theorem of Uniformisation ensures that every simply connected Riemann surface is isomorphic either to the plane, the sphere or the unit disc, and therefore these are the only possible universal covers.

If the genus of a compact Riemann surface is greater than or equal to 2, the universal cover is necessarily the disc, whose group of automorphisms is isomorphic to $\mathrm{PSL}(2, \mathbb{R})$. One of the main properties of this group of automorphisms is the fact that it agrees with the group of isometries of the disc with the hyperbolic metric, and therefore every Riemann surface of genus greater than or equal to two inherits a hyperbolic metric in a natural way. The subgroups of $\mathrm{PSL}(2, \mathbb{R})$ defining a Riemann surface in the quotient do not necessarily act freely, it is enough that they act properly discontinuously. Such groups are called Fuchsian groups.

Among Fuchsian groups, an important family is that of triangle groups, which are groups generated by rotations around the three vertices of a hyperbolic triangle and which define in the quotient a Riemann surface of genus 0 with three marked points. Triangle groups are closely related to both dessins d'enfants and Beauville surfaces, which are the principal objects of study of this thesis.

A dessin d'enfant is a finite bipartite graph on a compact orientable topological surface whose complement is a finite union of topological discs. Every dessin endows the topological surface in which it is embedded with a Riemann surface structure. Moreover, by the theorem of Belyi–Grothendieck, this Riemann surface corresponds to an algebraic curve with coefficients in the field $\overline{\mathbb{Q}}$ of algebraic numbers, and conversely, to every curve with algebraic coefficients one can associate at least one dessin.

On the other hand, a Beauville surface is a complex surface (real dimension 4) isomorphic to the quotient of the product of two compact Riemann surfaces by a certain free action of a finite group which acts by isomorphisms. This action must restrict to each of the two Riemann surfaces, defining in the quotient a surface of genus 0 with three marked points. The importance of these surfaces comes mainly from the fact that they are rigid, that is they do not admit non-trivial deformation.

Apart from the preliminaries chapter, in which the concepts necessary to develop the following chapters are presented, the rest of the text can be divided into three parts.

In Chapter 1 we define the triangle groups, which are at the core of the concepts which are introduced in Chapters 2 and 3, and we describe their properties. In particular we prove with elementary methods of hyperbolic geometry the well-known fact that every group isomorphism between triangle groups is induced by an isometry of the hyperbolic plane (Corollary 1.1). We study the relation between triangle groups, dessins d'enfants and triangle curves (or quasiplatonic curves) and describe the action of the absolute Galois group on triangle $G-$coverings from the point of view of their defining triples (Proposition 1.3), formalising a method by M. Streit. As a consequence we prove the already known fact that abelian triangle $G-$coverings can be defined over $\mathbb{Q}$ (Corollary 1.2). Finally, we characterize all the triangle coverings with group $\mathrm{PSL}(2,p)$ and of types (p,p,p) and $(2,3,n)$ (Theorems 1.2 and 1.3) and the triangle coverings with group $\mathrm{PSL}(2,7)$ and of type $(3,3,4)$ (Theorem 1.4).

In Chapter 2 we study the existence of multiple uniform dessins of the same type on a Riemann surface. In the non-arithmetic case the result is immediate (Theorem 2.1). In the case in which the group uniformising the surface is arithmetic, through the study of maximal orders in quaternion algebras we find a necessary and sufficient condition for a Riemann surface to contain different uniform dessins (Theorem 2.3). We also show several examples of well-known Riemann surfaces in which we prove, using the previous results, that different uniform dessins of the same type live (Section 2.4). Lastly we explain another method to find multiple uniform dessins on the same surface, which leads to the characterization of unicellular uniform dessins in genus 2 (Theorem 2.5).

In Chapter 3 we introduce the concept of a Beauville surface and enumerate the properties of these complex surfaces. In the first sections we present the theory of these surfaces from the point of view of uniformisation, proving results of Catanese from another perspective (Sections 3.1 and 3.2). In these sections there are also some new results, for example, restrictions to the genera that the curves defining a Beauville surface can have (Proposition 3.1 and Theorem 3.1 in the unmixed case, and Corollary 3.4 in the mixed case) or considerations about the the groups of automorphisms of a Beauville surface (Theorem 3.2 in the unmixed case and Theorem 3.3 in the mixed case). In Section 3.3 we prove results equivalent to Catanese's rigidity theorems for Beauville surfaces from the point of view of Fuchsian groups (Theorem 3.4). Finally, we construct several examples of non-homeomorphic Beauville surfaces which are Galois conjugate: first the example with minimum genera and Beauville group $\mathrm{PSL}(2,7)$ (Theorem 3.6 and Corollary 3.7) and then an infinite family of Beauville surfaces with Beauville group $\mathrm{PSL}(2,p)$ (Theorems 3.7 and 3.8).

The main results of this thesis can be found in the following papers:

- E. Girondo, D. Torres-Teigell: *Genus 2 Belyĭ surfaces with a unicellular uniform dessin*, Geom. Dedicata **155** (2011), 81–103.
- E. Girondo, D. Torres-Teigell, J. Wolfart: *Shimura curves with many uniform dessins*, Math. Z. (2011), doi:10.1007/s00209-011-0889-4.

- G. González-Diez, D. Torres-Teigell: *An introduction to Beauville surfaces via uniformization*, in "Quasiconformal Mappings, Riemann Surfaces, and Teichmüller Spaces", Contemp. Math. (2012), in press.
- G. González-Diez, D. Torres-Teigell: *Non-homeomorphic Galois conjugate Beauville structures on* $\mathrm{PSL}(2,p)$, Adv. Math. (2012), doi:10.1016/j.aim.2012.02.014.

The following papers also include results related to the contents of this thesis:

- G. González-Diez, G. A. Jones, D. Torres-Teigell: *Beauville surfaces with abelian Beauville group*, arXiv:1102.4552v1.
- G. González-Diez, G. A. Jones, D. Torres-Teigell: *Arbitrarily large Galois orbits of non-homeomorphic surfaces*, arXiv:1110.4930.

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— J. R. R. Tolkien, The Fellowship of the Ring

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Contents

Common notations

$\overline{\mathbb{Q}}$	field of algebraic numbers
$\mathbb{F}_q$	finite field with q elements
$\mathbb{H}$	half-plane model of the hyperbolic plane
$\mathbb{D}$	unit disc model of the hyperbolic plane
$\mathbb{S}^2$	unit sphere
$\widehat{\mathbb{C}}$	complex extended plane
$\mathbb{P}^1(\mathbb{C})$	complex projective line
k_v	local field of k with respect to a valuation v
R_k	ring of integers of the field k
A^*	group of invertible elements in A
A^1	group of elements of norm 1 in A
$M_2(R)$	ring of 2×2 matrices over the ring (or field) R
$\mathrm{GL}(2, R)$	group of invertible 2×2 matrices over the ring (or field) R
$\mathrm{SL}(2, R)$	group of unit 2×2 matrices over the ring (or field) R
$\Delta(l, m, n)$	hyperbolic triangle group of type (l, m, n)
$C_H(g)$	centraliser of the element $g \in G$ in the subgroup $H < G$
$\mathrm{Comm}(H)$	commensurator of the subgroup $H < G$
$k^G = \mathrm{Fix}(G)$	subfield of k fixed by the subgroup $G < \mathrm{Gal}(k/\mathbb{Q})$

Preliminaries

In this chapter we introduce many basic notions which will be used later in the rest of the chapters. Its content is mostly well known and the results are stated without proof.

In section 0.1 we give an introduction to Riemann surfaces. Although there is a huge amount of literature on this subject, perhaps the most suitable references are [**54**, **22**].

In sections 0.2 and 0.3 we present two important parts of the theory of compact Riemann surfaces: Galois actions on algebraic curves and covering theory of Riemann surfaces. For our point of view the most appropriate reference is probably [**32**].

Sections 0.4 and 0.5 deal with Fuchsian groups, their fundamental domains and their relation with hyperbolic geometry. Most of what is presented here can be found in [**10**, **48**].

In section 0.6 we present the Grothendieck–Belyi theory of dessins d'enfants and Belyi functions. We refer the reader to [**32**] for a comprehensive and more formal exposition (see also [**56**]).

Finally, in sections 0.7 and 0.8 we introduce some notions from the theory of quaternion algebras. For an exhaustive introduction to this subject see for example [**65**, **51**] (see also [**50**, **49**]).

0.1. Riemann surfaces

A Riemann surface is a topological surface with a complex structure, i.e. with an atlas $\{(U_i, \varphi_i)\}$ such that the transition functions $\varphi_i \circ \varphi_j^{-1}$ are holomorphic functions between open sets of the complex plane $\mathbb{C}$. By the Cauchy–Riemann equations, every Riemann surface is orientable, and therefore the compact ones are topologically characterized by their genus.

The most basic examples of Riemann surfaces are open sets of the complex plane $U \subset \mathbb{C}$ with the identity atlas $\{(U, \mathrm{Id})\}$. In particular one has the complex plane $\mathbb{C}$, the upper half-plane $\mathbb{H} = \{\mathrm{w} \in \mathbb{C} \;:\; \mathrm{Im(w)} > 0\}$ and the unit disc $\mathbb{D} = \{\mathrm{w} \in \mathbb{C} \;:\; |\mathrm{w}| < 1\}$. Other surfaces that can be given a Riemann surface structure are the unit sphere $\mathbb{S}^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$, the complex extended plane (or Riemann sphere) $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ and the complex projective line $\mathbb{P}^1 := \mathbb{P}^1(\mathbb{C})$.

It is because of the complex structure that one can define in a natural way holomorphic and meromorphic functions on Riemann surfaces and morphisms between them. The following Riemann surfaces are isomorphic: $\mathbb{H} \cong \mathbb{D}$ and $\mathbb{S}^2 \cong \widehat{\mathbb{C}} \cong \mathbb{P}^1$. In fact, these two are, together with the complex plane $\mathbb{C}$ the only simply connected Riemann surfaces.

THEOREM (Uniformisation theorem). *Any simply connected Riemann surface is isomorphic to $\mathbb{D}$, $\mathbb{C}$ or $\widehat{\mathbb{C}}$.*

It is a classical fact that there exists a bijection between isomorphism classes of compact Riemann surfaces and isomorphism classes of non-singular projective algebraic curves over the complex field. We will therefore use interchangeably the terms Riemann surface and algebraic curve.

Let now $f : S \longrightarrow \mathbb{C}$ be a meromorphic function on the Riemann surface S and let $P \in S$ be a zero or a pole of f. We define the *order* of f in P as the order of the local expression $f \circ \varphi^{-1}$ in $\varphi(P)$ and it is denoted by $\mathrm{ord}_P(f)$. If $f : S \longrightarrow S'$ is a morphism and we choose a local expression of f such that $(\psi \circ f \circ \varphi^{-1})(\varphi(P)) = 0$, the *multiplicity* of f in P is defined as $m_P(f) := \mathrm{ord}_{\varphi(P)}(\psi \circ f \circ \varphi^{-1})$. If $m_P(f) \geq 2$ we say that P is a *ramification point* and that $f(P)$ is a *ramification value* of f. For any non-constant morphism $f : S \longrightarrow S'$ of compact Riemann surfaces we can define the *degree* of f as $\deg(f) := \sum_{f(p)=y} m_p(f)$, which does not depend on the choice of $y \in S'$. Given a morphism f, the genera $g(S)$ and $g(S')$ of S and S' are related by the Riemann–Hurwitz formula:

$$2g(S) - 2 = \deg(f)(2g(S') - 2) + \sum_{p \in S} (m_p(f) - 1)\,.$$

As for automorphisms of compact Riemann surfaces, i.e. isomorphisms of S onto itself, there is a bound to the order of the automorphism group $\mathrm{Aut}(S)$ of S in terms of its genus $g(S)$. This bound, called Hurwitz bound, states that for $g(S) \geq 2$ one has $|\mathrm{Aut}(S)| \leq 84(g(S) - 1)$. The Riemann surfaces achieving it are called Hurwitz curves, and any finite group G which occurs as the full automorphism group of one of these surfaces is called a Hurwitz group.

0.2. Action of the Galois group

The Galois group $\mathrm{Gal}(\mathbb{C}) := \mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ acts naturally on complex algebraic varieties in the following way. Let first $S = \{[x, y, z] \in \mathbb{P}^2(\mathbb{C}) : F(x, y, z) = 0\}$ be a projective algebraic curve given as the zeroes of a homogeneous polynomial $F \in \mathbb{C}[X, Y, Z]$. If $\sigma \in \mathrm{Gal}(\mathbb{C})$ is a field automorphism of $\mathbb{C}$ one can construct the Galois conjugate curve $S_F^\sigma = S_{F^\sigma}$, where F^σ is obtained from F by applying σ to its coefficients. We can proceed in the same way in higher dimension (or if the model for the curve S is not plane), so that if $V = \{F_\alpha = 0\}$ is an algebraic variety defined as the set of zeroes of a finite collection of polynomials $\{F_\alpha\} \subset \mathbb{C}[X_1, \ldots, X_n]$, the Galois conjugate variety is defined as the set of zeroes $V^\sigma = \{F_\alpha^\sigma = 0\}$.

Let now S be a compact Riemann surface and $k \subseteq \mathbb{C}$ a field. We say that k is a *field of definition* of S if there exists a finite collection of homogenous polynomials $F \subset k[X_1, \ldots, X_n]$ such that S and $S_F = \{[x_1, \ldots, x_n] \in \mathbb{P}^{n-1}(\mathbb{C}) : Q(x_1, \ldots, x_n) = 0, \text{ for all } Q \in F\}$ are isomorphic. On the other hand if we define the inertia group

$$I_S = \{\sigma \in \mathrm{Gal}(\mathbb{C}) : S_F^\sigma \cong S_F\},$$

which clearly does not depend on the algebraic model of S, then the fixed field

$$\mathbb{C}^{I_S} = \mathrm{Fix}(I_S) = \{\alpha \in \mathbb{C} : \sigma(\alpha) = \alpha, \text{ for all } \sigma \in I_S\}$$

is called the *field of moduli* of S, and it is denoted by $M(S)$. In particular the index of I_S in $\mathrm{Gal}(\mathbb{C})$ agrees with the cardinality of the orbit of S under the action of $\mathrm{Gal}(\mathbb{C})$. The field of moduli of a Riemann surface is always contained in any field of definition, but the converse is not true in general, as shown by well-known counterexamples ([20, 59]). The concepts of field of definition and field of moduli of a complex algebraic variety V of arbitrary dimension can be defined in the same way.

Let us consider now an automorphism $\tau \in \mathrm{Aut}(S)$ of order r, and let $P \in S$ be a fixed point of τ. If ψ is a local parameter around P such that $\psi(P) = 0$, then locally

$$\psi \circ \tau \circ \psi^{-1}(\mathrm{w}) = \zeta_r^k \mathrm{w}, \text{ where } \zeta_r = e^{2\pi i/r} \text{ and } k \in \mathbb{Z}.$$

Since $\zeta_r^k = (\psi \circ \tau \circ \psi^{-1})'(0)$, it is clear that this root of unity does not depend on the local chart ψ chosen. We say then that τ rotates with angle $2\pi k/r$ in P, and that ζ_r^k (or simply k) is the *rotation number* of τ in P. Let us note that the rotation number k is defined modulo r.

The relevance of rotation numbers lies in the fact that if $\tau : S \longrightarrow S$ is a finite order automorphism of S fixing a point P with rotation angle ζ and σ is a field automorphism of $\mathbb{C}$ (or any other field of definition of S and τ), then $\tau^\sigma : S^\sigma \longrightarrow S^\sigma$ is a finite order automorphism fixing P^σ with rotation angle $\sigma(\zeta)$. The reason for this relation is the following: let $\tau^* : H^0(S, \Omega) \longrightarrow H^0(S, \Omega)$ be the $\mathbb{C}-$linear automorphism induced by τ in the space of regular $1-$forms and ω be an eigenvector with eigenvalue λ such that $\omega(P) \neq 0$. Such holomorphic $1-$form must exist because on the one hand τ^* admits a diagonal basis and on the other hand, by the Riemann–Roch theorem, not all holomorphic $1-$forms vanish simultaneously at a point P. Therefore in terms of a local coordinate around P we can write

$$\begin{aligned} \tau(\mathrm{w}) &= \zeta \mathrm{w}\,; \\ \omega &= (a_0 + a_1\mathrm{w} + a_2\mathrm{w}^2 + \ldots)d\mathrm{w}\ , \quad a_0 \neq 0; \quad \text{and} \\ \lambda\omega &= \tau^*\omega \ = \ (a_0 + a_1\zeta\mathrm{w} + a_2\zeta^2\mathrm{w}^2 + \ldots)\zeta d\mathrm{w}. \end{aligned}$$

One concludes that the rotation number ζ equals the eigenvalue λ – an algebraically defined object – and so $\sigma(\lambda) = \sigma(\zeta)$, which clearly is the eigenvalue corresponding to the eigenvector ω^σ of $(\tau^\sigma)^*$, must be the rotation number of the automorphism τ^σ at the point $P^\sigma \in S^\sigma$.

0.3. Uniformisation and universal coverings

The general theory of covering spaces tells us that any topological manifold X admits a simply connected *universal covering* $\widetilde{X}$. Furthermore, if X has a complex structure the universal cover can be endowed with a complex structure such that the projection $\widetilde{X} \longrightarrow X$ is a morphism.

In the particular case of surfaces, this theory ensures that any Riemann surface S can be written as the quotient $S = \widetilde{S}/G$ of a simply connected Riemann surface $\widetilde{S}$ by the free action of a subgroup G of the group of automorphisms $\mathrm{Aut}(\widetilde{S})$, which is moreover isomorphic to the fundamental group $\pi_1(S)$. In this case, the situation is quite easy since by the uniformisation theorem the universal covering of any Riemann surface S must be isomorphic either to $\mathbb{D}$, $\mathbb{C}$ or $\mathbb{P}^1(\mathbb{C})$. Now the only Riemann surface having $\mathbb{P}^1(\mathbb{C})$ as universal cover is precisely $\mathbb{P}^1(\mathbb{C})$, since any automorphism of $\mathbb{P}^1(\mathbb{C})$ has fixed points. As for the complex plane, one has $\mathrm{Aut}(\mathbb{C}) \cong \{z \longmapsto az + b \ : \ a, b \in \mathbb{C}\}$ and any subgroup $G < \mathrm{Aut}(\mathbb{C})$ which does not fix points is a group of translations, therefore abelian; hence no compact Riemann surface of genus greater than or equal to two can have $\mathbb{C}$ as universal covering, since its fundamental group is not abelian. As a first consequence, it becomes particularly important the group of automorphisms of the disc, since almost every compact Riemann surface will be uniformised by a torsion-free subgroup of it.

The groups of automorphisms of $\mathbb{H}$ and $\mathbb{D}$ are isomorphic to $\mathrm{PSL}(2,\mathbb{R})$ and they can be identified with

$$\begin{aligned} \mathrm{Aut}(\mathbb{H}) &= \left\{ \mathrm{w} \longmapsto \frac{a\mathrm{w}+b}{c\mathrm{w}+d} \;:\; a,b,c,d \in \mathbb{R}, ad-bc \neq 0 \right\} \quad \text{and} \\ \mathrm{Aut}(\mathbb{D}) &= \left\{ \mathrm{w} \longmapsto e^{i\theta} \frac{\mathrm{w}-\alpha}{1-\overline{\alpha}\mathrm{w}} \;:\; \alpha \in \mathbb{D}, \theta \in \mathbb{R} \right\}. \end{aligned}$$

This fact can be generalised to higher dimensions. The following result follows from a theorem of Cartan ([36], see also [55]):

Proposition 0.1. *Let* $f \in \mathrm{Aut}(\mathbb{H}\times\mathbb{H})$. *There exist* $\tilde{f}_1, \tilde{f}_2 \in \mathrm{PSL}(2,\mathbb{R})$ *such that*

$$\tilde{f}(\mathrm{w}_1,\mathrm{w}_2) = \begin{cases} (\tilde{f}_1(\mathrm{w}_1), \tilde{f}_2(\mathrm{w}_2)), & \text{if } \tilde{f} \text{ is factor-preserving,} \\ (\tilde{f}_1(\mathrm{w}_2), \tilde{f}_2(\mathrm{w}_1)), & \text{if } \tilde{f} \text{ is factor-reversing.} \end{cases}$$

In particular, $\mathrm{Aut}(\mathbb{H}\times\mathbb{H}) = (\mathrm{Aut}(\mathbb{H})\times\mathrm{Aut}(\mathbb{H})) \rtimes \langle J\rangle$, *where* $\langle J\rangle$ *is the group of order two generated by the automorphism* $J(\mathrm{w}_1,\mathrm{w}_2) = (\mathrm{w}_2,\mathrm{w}_1)$.

Finally, the genus of a compact Riemann surface determines its universal covering.

Proposition 0.2. *Compact Riemann surfaces can be characterized in the following way:*

(i) *the only compact Riemann surface of genus zero is the Riemann sphere* $\mathbb{P}^1(\mathbb{C})$;

(ii) *the universal covering of any compact Riemann surface of genus one is the complex plane* $\mathbb{C}$, *and the group of deck transformations is a lattice:*

$$\Lambda = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2, \qquad \text{with } \omega_1,\omega_2 \in \mathbb{C}, \text{ and } \tfrac{\omega_1}{\omega_2} \notin \mathbb{R};$$

(iii) *the universal covering of any compact Riemann surface of genus greater than or equal to two is the upper half-plane* $\mathbb{H}$, *and the group of deck transformations is a subgroup* $\Gamma < \mathrm{PSL}(2,\mathbb{R})$.

0.4. Fuchsian groups

The subgroups of $\mathrm{Aut}(\mathbb{H})$ which define a Riemann surface structure on the quotient do not necessarily act without fixed points. A *Fuchsian group* is a subgroup $\Gamma < \mathrm{PSL}(2,\mathbb{R})$ which is discrete with respect to the topology induced by the usual topology in $\mathbb{R}^4$. Fuchsian groups were introduced by Henri Poincaré in 1880 following writings of Lazarus Fuchs about differential equations. One can prove that a subgroup $\Gamma < \mathrm{PSL}(2,\mathbb{R})$ is a Fuchsian group if and only if it acts discontinuously on $\mathbb{H}$, i.e.

(i) Every $\mathrm{w} \in \mathbb{H}$ is a fixed point of only a finite number of transformations $\gamma_1 = \mathrm{Id}, \ldots, \gamma_r \in \Gamma$;

(ii) For every $\mathrm{w} \in \mathbb{H}$ there exists a neighbourhood U such that $\gamma(U) \cap U = \emptyset$ for every $\gamma \in \Gamma \setminus \{\gamma_1, \ldots, \gamma_r\}$.

The quotient $\mathbb{H}/\Gamma$ of $\mathbb{H}$ by the action of a Fuchsian group Γ has a natural Riemann surface structure. The elements of Γ that fix points in $\mathbb{H}$ correspond precisely those of finite order. If the resulting Riemann surface $\mathbb{H}/\Gamma$ is compact, the set of conjugacy classes of finite order elements of Γ is finite. One can take suitable representatives γ_i of order m_i such that for every $\mathrm{w} \in \mathbb{H}$ the set of elements of Γ fixing it is either trivial or a cyclic group generated by an element conjugate to one of the γ_i. Under these assumptions, if the Riemann surface defined by Γ has genus g we say that Γ has *signature* $(g; m_1, \dots, m_k)$.

Let now $\Gamma, \Gamma' < \mathrm{PSL}(2,\mathbb{R})$ be Fuchsian groups acting without fixed points on $\mathbb{H}$ and $S = \mathbb{H}/\Gamma$ and $S' = \mathbb{H}/\Gamma'$ be the corresponding (not necessarily compact) Riemann surfaces uniformised by them. Then S and S' are isomorphic if and only if there exists $\gamma \in \mathrm{PSL}(2,\mathbb{R})$ such that $\Gamma' = \gamma\Gamma\gamma^{-1}$. Moreover $\mathrm{Aut}(\mathbb{H}/\Gamma) \cong N(\Gamma)/\Gamma$, where $N(\Gamma) = \{\gamma \in \mathrm{PSL}(2,\mathbb{R}) \, : \, \gamma\Gamma\gamma^{-1} = \Gamma\}$ is the normaliser of Γ in $\mathrm{PSL}(2,\mathbb{R})$.

If Γ is not cyclic then $N(\Gamma)$ is also a Fuchsian group, and therefore compact Riemann surfaces of genus greater than or equal to two have finite group of automorphisms.

0.5. Hyperbolic geometry and fundamental domains

One of the most relevant facts about holomorphic self-mappings of the disc is their relation with hyperbolic geometry. Let us first recall some concepts about this geometry. The basic idea behind hyperbolic (plane) geometry is replacing Euclid's fifth postulate (more precisely Playfair's axiom):

> *For any given line L and point P not on L, there is exactly one line through P that does not intersect L.*

by the following one:

> *For any given line L and point P not on L, there are infinitely many lines through P that do not intersect L.*

The *hyperbolic plane* satisfies this new axiom. It is a simply connected Riemannian manifold of dimension 2 whose metric has constant curvature -1. The metrics

$$ds^2_{\mathbb{H}} = \frac{dx^2 + dy^2}{y^2} \qquad \text{and} \qquad ds^2_{\mathbb{D}} = \frac{dx^2 + dy^2}{(1 - (x^2 + y^2))^2}$$

on the upper half-plane $\mathbb{H}$ and on the disc $\mathbb{D}$ respectively turn them into models of the hyperbolic plane. These metrics are conformal to the Euclidean one in $\mathbb{R}^2$, and therefore the Euclidean angles are preserved.

One can compute the hyperbolic length of a curve $\gamma(t) = (x(t), y(t))$ and the hyperbolic area of a set E contained in $\mathbb{H}$ or in $\mathbb{D}$ through the formulae

$$\ell_{\mathbb{H}}(\gamma) = \int \frac{\sqrt{x'(t)^2 + y'(t)^2}}{y(t)} dt \,, \qquad A_{\mathbb{H}}(E) = \int\int_E \frac{dxdy}{y^2} \,,$$

$$\ell_{\mathbb{D}}(\gamma) = \int \frac{\sqrt{x'(t)^2 + y'(t)^2}}{1 - (x(t)^2 + y(t)^2)} dt \,, \qquad A_{\mathbb{D}}(E) = \int\int_E \frac{dxdy}{(1 - (x^2 + y^2))^2} \,.$$

In both models the geodesics of the hyperbolic metric are arcs of (generalised) circumferences which intersect perpendicularly the border, $\partial\mathbb{H} = \mathbb{R} \cup \{\infty\}$ in the case of $\mathbb{H}$ and $\partial\mathbb{D} = \mathbb{S}^1$ in the case of $\mathbb{D}$.

The group $\mathrm{Aut}(\mathbb{H}) \cong \mathrm{PSL}(2,\mathbb{R})$ of holomorphic self-mappings of $\mathbb{H}$ coincides with the group of orientation-preserving isometries of the hyperbolic metric and acts transitively on the set of hyperbolic geodesics. In particular, its elements preserve both hyperbolic distance and hyperbolic area.

Let us now consider a Fuchsian group $\Gamma < \mathrm{PSL}(2,\mathbb{R})$ acting on the upper-half plane. We will call *fundamental domain* of Γ to any closed subset $\Omega \subset \mathbb{H}$ such that:

(i) Ω contains at least one point of each orbit of Γ;
(ii) the interior of Ω does not contain points equivalent under Γ;
(iii) $A_{\mathbb{H}}(\partial\Omega) = 0$, where $\partial\Omega$ is the border of Ω.

If Ω is a fundamental domain, $\mathbb{H} = \bigcup_{\gamma\in\Gamma} \gamma(\Omega)$ and we say that Ω and its images under Γ form a *tessellation* of $\mathbb{H}$. There is a specific kind of fundamental domains with particularly nice properties. Let p be a point not fixed by any non-trivial element of Γ. We call *Dirichlet region* of Γ centered at p to the set

$$D_p(\Gamma) = \{\mathrm{w} \in \mathbb{H} \ : \ \rho_{\mathbb{H}}(\mathrm{w}, p) \leq \rho_{\mathbb{H}}(\gamma(\mathrm{w}), p), \ \forall \gamma \in \Gamma\},$$

where $\rho_{\mathbb{H}}$ is the hyperbolic distance. The region $D_p(\Gamma)$ is an intersection of hyperbolic half-planes and therefore it is a convex hyperbolic polygon, i.e. a closed connected set on $\overline{\mathbb{H}}$ whose border is formed by arcs of hyperbolic geodesics. As a consequence one can represent the compact Riemann surface $\mathbb{H}/\Gamma$ as a fundamental polygon $\mathcal{P}$ together with a side pairing on the sides $s_1, \ldots, s_n$, so that for every s_i there is an $s_{j(i)}$ and a $\gamma \in \Gamma$ such that $\gamma(s_i) = s_{j(i)}$.

Moreover, if the group of elements of Γ fixing a vertex $v_j \in \mathcal{P}$ is generated by $\gamma_j \in \Gamma$, then the angle at v_j is $\alpha_j = 2\pi/\mathrm{ord}(\gamma_j)$. The converse is included in the following theorem.

Theorem 0.1 (Poincaré). *Let $\mathcal{P} \subset \overline{\mathbb{H}}$ be a hyperbolic polygon with (unordered) sides $s_1, \ldots, s_n, s'_1, \ldots, s'_n$. Suppose that there exist elements $\gamma_i \in \mathrm{PSL}(2,\mathbb{R})$ such that $\gamma_i(s_i) = s'_i$ for each $i = 1, \ldots, n$ and let $\Gamma = \langle \gamma_1, \ldots, \gamma_n \rangle$. If for any complete collection V_j of vertices of $\mathcal{P}$ equivalent under Γ the sum of its angles is equal to $2\pi/m_j$ with $m_j \in \mathbb{N}$, then the group Γ acts properly discontinuously on $\mathbb{H}$ and $\mathbb{H}/\Gamma$ is a Riemann surface. If moreover $\mathcal{P} \cap \partial\mathbb{H} = \emptyset$, then $\mathbb{H}/\Gamma$ is compact.*

0.6. Dessins d'enfants

In the Grothendieck-Belyi theory of dessins d'enfants there are two main ingredients. First, a *dessin d'enfant* is a pair $(S, \mathcal{D})$, where S is a compact oriented topological surface and $\mathcal{D}$ is a finite graph embedded in S satisfying the following properties:

(i) it is a bicoloured graph, i.e. every vertex has an assigned colour, white ($\circ$) or black ($\bullet$), in such a way that the two vertices of an edge have always different colours;
(ii) each connected component of the complement $S \setminus \mathcal{D}$ is homeomorphic to a disc. Each of them will be called face of the dessin.

We will regard two dessins $(S_1, \mathcal{D}_1)$ and $(S_2, \mathcal{D}_2)$ as equivalent (or isomorphic) if there exists an orientation-preserving homeomorphism $f : S_1 \longrightarrow S_2$ whose restriction $f|_{\mathcal{D}_1}$ induces an isomorphism of bicoloured graphs $f|_{\mathcal{D}_1} : \mathcal{D}_1 \longrightarrow \mathcal{D}_2$. The degree of a vertex of $\mathcal{D}$ is defined as the number of incident edges and the degree of a face is defined as half the number of edges delimiting that face, counting multiplicities. If the least common multiples of the degrees of the white vertices, black vertices and faces are l, m and n respectively, we will say that the type of the dessin is (l, m, n).

The other ingredient is Belyi functions. A *Belyi function* is a meromorphic function $\beta : S \longrightarrow \mathbb{P}^1$ on a Riemann surface S with three ramification values at most, which we can suppose to be 0, 1 and ∞. We will consider two Belyi pairs (S, f) and (S', f') equivalent if there exists an isomorphism $F : S \longrightarrow S'$ such that $f = f' \circ F$.

Grothendieck pointed out that there is a bijective correspondence between equivalence classes of dessins d'enfants and equivalence classes of Belyi pairs. To recover a dessin from a Belyi function β one simply takes the inverse image of the interval $[0, 1]$ under β and considers $\beta^{-1}(0)$ as white vertices and $\beta^{-1}(1)$ as black vertices. Constructing a Belyi function from a dessin $\mathcal{D}$ is slightly more complicated. It can be achieved by considering a triangulation associated to $\mathcal{D}$ and constructing a topological covering β from S minus the set of vertices and face centres of $\mathcal{D}$ to $\mathbb{P}^1 \setminus \{0, 1, \infty\}$, which endows S with a Riemann surface structure $S_{\mathcal{D}}$ to which β extends as a meromorphic function with three ramification values. The degree of a given white vertex, black vertex or face of the dessin can be understood then as the ramification order of β in such point.

The importance of this fact lies on its relation with the theorem of Belyi ([12]), which states that a compact Riemann surface S is isomorphic to an algebraic curve defined over the field of algebraic numbers $\overline{\mathbb{Q}}$ if and only if there exists a Belyi function $f : S \longrightarrow \mathbb{P}^1$.

The fact that any Riemann surface admitting a Belyi function can be defined over $\overline{\mathbb{Q}}$ was already known and it follows from Weil's criterion ([67], see also [37]). However the proof of the other implication, which is due to Belyi, is as astonishing as simple. Grothendieck himself wrote about it in [44]: *"[...]Belyi annonce justement ce résultat, avec une démonstration d'une simplicité déconcertante tenant en deux petites pages d'une lettre de Deligne – jamais sans doute un résultat profond et déroutant ne fut démontré en si peu de lignes!"* [1]. This proof is based on constructing a function f from S to the sphere $\mathbb{P}^1$ ramified only over rational values, and compose it with suitable Belyi polynomials, which are polynomials of the form

$$P_{m,n}(\mathrm{w}) = \frac{(m+n)^{m+n}}{m^m \cdot n^n} \mathrm{w}^m (1-\mathrm{w})^n.$$

The relevant fact is that 0, 1, $\frac{m}{m+n}$ and ∞ are the only ramification points of this polynomial, and they are sent to $\{0, 1, \infty\}$. Therefore, one can compose the function f with consecutive suitable polynomials P_{m_i,n_i} so that so that the set of ramification values of the resulting function ends up being the set $\{0, 1, \infty\}$.

THEOREM (Belyi–Grothendieck). *For any Riemann surface S of genus g defined over $\overline{\mathbb{Q}}$ there exists a dessin $\mathcal{D}$ on the compact oriented topological surface of genus g such that $S = S_{\mathcal{D}}$.*

0.7. Quaternion algebras

The following definition works for any field k, but we will only focus on subfields of the complex field and its localisations (some of the following statements might not be true in characteristic 2). A *quaternion algebra* A over k is a 4-dimensional

[1] The translation into English that can be found in the introduction of [56] reads: *"[...]Belyi announced exactly that result, with a proof of a disconcerting simplicity which fit into two little pages of a letter of Deligne – never, without a doubt, was such a deep and disconcerting result proved in so few lines!"*.

central simple $k-$algebra, i.e. a $k-$algebra of dimension 4 without proper two-sided ideals, whose centre agrees with the field k. There always exist $a, b \in k^*$ and a basis $\{1, i, j, ij\}$ of A such that we can write

$$A = \{x_0 + x_1 i + x_2 j + x_3 ij : x_0, x_1, x_2, x_3 \in k, i^2 = a, j^2 = b, ij = -ji\}.$$

Note that $(ij)^2 = -ab$. Conversely, any choice of $a, b \in k^*$ defines a quaternion algebra A over k. Under these conditions we will denote it by the Hilbert symbol $A = (\frac{a,b}{k})$. However different choices of a and b can lead to isomorphic algebras.

Given an element $x = x_0 + x_1 i + x_2 j + x_3 ij$, its conjugate is defined as $\overline{x} = x_0 - x_1 i - x_2 j - x_3 ij$. This allows us to define a *reduced norm* and a *reduced trace* on A as

$$n(x) = x\overline{x} = x_0^2 - ax_1^2 - bx_2^2 + abx_3^2, \quad \text{and} \quad \mathrm{tr}(x) = x + \overline{x} = 2x_0.$$

All these definitions do not depend on the choice of basis. The invertible elements of A, the set of which is denoted by A^*, are precisely those x such that $n(x) \neq 0$. We will write $A^1 \subset A^*$ for the subgroup of elements of norm 1.

We will also need the following well-known theorem ([65], Ch. I, Thm 2.1).

Theorem (Skolem–Noether). *Let $M, M' \subset A$ be subalgebras of a quaternion $k-$algebra A. Any isomorphism of algebras $\phi : M \longrightarrow M'$ is induced by an inner automorphism of A.*

It is known that any quaternion $k-$algebra A is either a division algebra or isomorphic to $M_2(k)$ (if k is algebraically closed, it is necessarily isomorphic to $M_2(k)$). In the case when $A = (\frac{a,b}{k})$ is a division algebra, there exists an injection

$$\rho : A \longrightarrow M_2(k(\sqrt{a})) \tag{0.1}$$

of A into the 2×2 matrices over the quadratic field extension $k(\sqrt{a})$ of k, so one can always regard any quaternion algebra as an algebra of matrices. Moreover, via this identification the reduced norm $n(x)$ and the reduced trace $\mathrm{tr}(x)$ on A coincide with the matrix determinant $\det(\rho(x))$ and the matrix trace $\mathrm{tr}(\rho(x))$.

The easiest examples of quaternion algebras are the Hamilton quaternions $\mathcal{H} = (\frac{-1,-1}{\mathbb{R}})$ and the algebra of matrices $M_2(\mathbb{R}) = (\frac{1,1}{\mathbb{R}})$ and, in fact, these two are the only quaternion algebras over the real field, up to isomorphism. An analogous situation occurs over $p-$adic fields. Any quaternion algebra over a $p-$adic field k_p is isomorphic either to $M_2(k_p)$, or to a unique division algebra.

Let L/k be a field extension. If $A = (\frac{a,b}{k})$ we can define the quaternion $L-$algebra $A \otimes_k L = (\frac{a,b}{L})$. In particular, for any valuation v on k we can define the local quaternion algebra $A_v = A \otimes_k k_v$, where k_v is the localisation of k with respect to v. If A_v is isomorphic to $M_2(k_v)$ we say that A splits at the valuation v; otherwise we say that A ramifies at v. We will write $\Omega_f(k)$ for the set of non-Archimedean valuations of k, and the subset of $\Omega_f(k)$ consisting of the valuations at which A ramifies is denoted by $\mathrm{Ram}_f(A)$. The valuations $v \in \mathrm{Ram}_f(A)$ correspond to certain primes ideals $\mathfrak{P}$, and the *discriminant* of A is defined as $D(A) = \prod_{v \in \mathrm{Ram}_f(A)} \mathfrak{P}$.

Similarly, for any Galois element $\sigma \in \mathrm{Gal}(\mathbb{C})$ one can define the quaternion $\sigma(k)-$algebra $A^\sigma = (\frac{\sigma(a),\sigma(b)}{\sigma(k)})$.

Let now R_k be the ring of integers of k. An *order* in the quaternion $k-$algebra A is an R_k-module $\mathcal{O} \subset A$ which is a ring with unity and such that $\mathcal{O} \otimes_{R_k} k = A$. We say that $\mathcal{O}$ is a *maximal order* if it is maximal with respect to the inclusion and we call it an *Eichler order* if it is the intersection of two maximal orders ([65], p. 20). As before, given an order $\mathcal{O}$ in A, for each valuation v on k we can define the order $\mathcal{O}_v = \mathcal{O} \otimes_{R_k} R_v$, where R_v is the ring of integers of the local field k_v. One has the following result (see for example [51], Lemma 6.2.7).

LEMMA 0.1. *Fix an order $\mathcal{I}$ in A. Given any other order $\mathcal{O}$ in A, for almost every non-Archimedean valuation v one has $\mathcal{I}_v = \mathcal{O}_v$. Moreover, there is a bijection*

$$\{\text{orders } \mathcal{O} \subset A\} \quad \longleftrightarrow \quad \{(\mathcal{L}_v)_{v\in\Omega_f(k)} \ : \ \begin{array}{l}\mathcal{L}_v \text{ is an order in } A_v, \\ \mathcal{L}_v = \mathcal{I}_v \text{ for almost all } v \in \Omega_f(k)\}\end{array}$$

given by $\mathcal{O} \longmapsto (\mathcal{O}_v)_{v\in\Omega_f(k)}$. This bijection preserves inclusion.

In the case where $A = M_2(k)$ and R_k is a principal ideal domain, all maximal orders are conjugate in A to $M_2(R_k)$. In general the number of conjugacy classes of maximal orders of A is called the *type number* of A.

The $p-$adic situation is simpler. If k_p is a finite extension of the $p-$adic field $\mathbb{Q}_p$, then the ring of integers R_p has only one maximal ideal $\mathcal{P}$. If A is a division algebra over k_p, then it has only one maximal order. On the other hand, if A is isomorphic to $M_2(k_p)$, then its maximal orders are usually represented as vertices of a regular tree of valency $q+1$, where the norm q denotes the number of elements of the residue class field $R_p/\mathcal{P}$ (see [65] pp. 40–41). Two vertices are joined by an edge if and only if the corresponding maximal orders are conjugate by an element whose norm is in $R_p^*\mathcal{P}$ (see Figure 0.1).

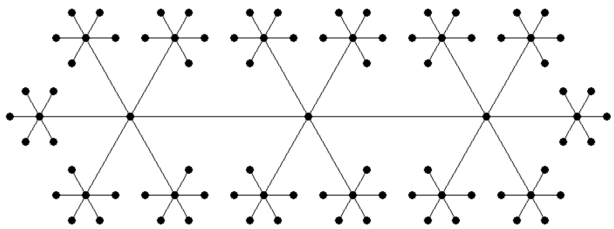


FIGURE 0.1. Part of the tree of local maximal orders for $q = 5$.

0.8. Arithmetic Fuchsian groups

Let Γ_1, Γ_2 be Fuchsian groups. They are said to be *commensurable* if their intersection has finite index in both of them, i.e. $[\Gamma_1 : \Gamma_1 \cap \Gamma_2] < \infty$ and $[\Gamma_2 : \Gamma_1 \cap \Gamma_2] < \infty$. The *commensurator group* of a Fuchsian group Γ is then defined as

$$\mathrm{Comm}(\Gamma) = \{\gamma \in \mathrm{PSL}(2,\mathbb{R}) : [\Gamma : \Gamma \cap \gamma\Gamma\gamma^{-1}] < \infty \text{ and } [\gamma\Gamma\gamma^{-1} : \Gamma \cap \gamma\Gamma\gamma^{-1}] < \infty\}.$$

We define the *invariant trace field* of Γ as the field $k_\Gamma = \mathbb{Q}(\mathrm{tr}(\Gamma^2))$ generated by the traces of the squares of elements of Γ. This is an invariant of the commensurability class of Γ, in other words any other Fuchsian group commensurable with Γ has the same invariant trace field.

The following is a consequence of a more general theorem by Borel and Harish-Chandra. Let k be a totally real number field, i.e. a number field all of whose embeddings $\sigma(k) \subset \mathbb{C}$ lie in $\mathbb{R}$. Let A be a quaternion algebra over k ramified at all infinite valuations but one, that is, such that $A \otimes_k \mathbb{R} \cong M_2(\mathbb{R})$ and for every $\sigma \in \mathrm{Gal}(\mathbb{C})$ with $\sigma \neq \mathrm{id}$ one has $A^\sigma \otimes_{\sigma(k)} \mathbb{R} \cong \mathcal{H}$. Note that under these conditions, the injection ρ in equation (0.1) allows us to regard A as a subalgebra of $M_2(\mathbb{R})$. Let $\mathcal{O}$ be an order in A and write $\mathcal{O}^1$ for its norm 1 group. Then the subgroup $\mathrm{P}\rho(\mathcal{O}^1) \subset \mathrm{PSL}(2,\mathbb{R})$, where the P stands for the usual projection $\mathrm{SL}(2,\mathbb{R}) \to \mathrm{PSL}(2,\mathbb{R})$, is a Fuchsian group.

A Fuchsian group Γ is said to be an *arithmetic Fuchsian group* if it is commensurable with any such $\mathrm{P}\rho(\mathcal{O}^1)$.

The most classical example of an arithmetic Fuchsian group is $\mathrm{PSL}(2,\mathbb{Z})$. It is (the projective image of) the norm 1 group of the ring of matrices $M_2(\mathbb{Z})$, which is an order in the quaternion $\mathbb{Q}-$algebra $M_2(\mathbb{Q})$. Note that this quaternion algebra trivially satisfies the hypothesis above.

Very few among all Fuchsian groups are arithmetic, but they play a central role in many situations. One of the points in which they differ from the non-arithmetic Fuchsian groups is the following ([**52**]).

THEOREM (Margulis). *Let* Γ *be a Fuchsian group. Then* Γ *is non-arithmetic if and only if* $\mathrm{Comm}(\Gamma)$ *is an extension of finite index of* Γ. *Otherwise* $\mathrm{Comm}(\Gamma)$ *is dense in* $\mathrm{PSL}(2,\mathbb{R})$.

CHAPTER 1

Triangle groups

"À bas Euclide! Mort aux triangles!"
— Jean Dieudonné

Triangle groups play a central role both in the theories of dessins d'enfants and of Beauville surfaces. They are Fuchsian groups which correspond to orbifolds of genus zero with three cone points.

More precisely, to construct a triangle group of signature (l, m, n) one considers a hyperbolic triangle T in the hyperbolic plane, with vertices v_0, v_1 and v_∞ and angles π/l, π/m and π/n respectively. The reflection R_i over the edge of T opposite to v_i is an anticonformal isometry of the hyperbolic plane. The group generated by these reflections acts discontinuously on $\mathbb{H}$ in such a way that T is a fundamental domain. The index-2 subgroup formed by the orientation-preserving transformations is called a triangle group of type (l, m, n). Elementary hyperbolic geometry ensures that the triangle T, and hence the corresponding triangle group, are unique up to conjugation in $\mathrm{PSL}(2, \mathbb{R})$. From now on we reserve the notation $T = T(l, m, n)$ for the triangle in the upper half-plane $\mathbb{H}$ which is the image under $M(\mathrm{w}) = \frac{i(1+\mathrm{w})}{1-\mathrm{w}}$ of the triangle depicted in Figure 1.1 inside the unit disc, i.e. the only triangle in $\mathbb{D}$ with $v_0 = 0$, $v_\infty \in \mathbb{R}^+$ and $v_1 \in \mathbb{D}^-$, the lower half-disc. The corresponding triangle group will be denoted by $\Delta = \Delta(l, m, n)$. Moreover, we will always place coinciding orders at the beginning of the triple, so that if two of them coincide, our triple will be (l, l, n). If the integers are all different we will always consider the triple (l, m, n) such that $l < m < n$.

The quadrilateral consisting of the union of T and one of its reflections $R_i(T)$ (e.g. the shaded triangle in the figure) serves as a fundamental domain for the group $\Delta(l, m, n)$, and therefore its images under $\Delta(l, m, n)$ tessellate the whole hyperbolic plane. Thus, the quotient $\mathbb{H}/\Delta$ is an orbifold of genus zero with three cone points $[v_0]_\Delta$, $[v_1]_\Delta$ and $[v_\infty]_\Delta$ of orders l, m and n respectively, where for an arbitrary Fuchsian group Λ the notation $[v]_\Lambda$ stands for the orbit of the point v under the action of Λ.

1.1. Properties of triangle groups

It is a classical fact that $\Delta(l, m, n)$ has presentation

$$\Delta(l, m, n) = \langle x, y, z \;:\; x^l = y^m = z^n = xyz = 1 \rangle \,,$$

where

$$x = R_1 R_\infty \,, \quad y = R_\infty R_0 \,, \quad z = R_0 R_1 \,, \tag{1.1}$$

are positive rotations around v_0, v_1 and v_∞ through angles $2\pi/l$, $2\pi/m$ and $2\pi/n$ respectively. It is also classical that any other finite order element of $\Delta(l,m,n)$ is conjugate to a power of x, y or z and that these account for all elements in Δ that fix points. We will always identify $\mathbb{H}/\Delta$ with $\mathbb{P}^1$ via the unique isomorphism

$$\begin{array}{rccc} \Phi: & \mathbb{H}/\Delta & \longrightarrow & \mathbb{P}^1 \\ & [v_0]_\Delta & \longmapsto & 0 \\ & [v_1]_\Delta & \longmapsto & 1 \\ & [v_\infty]_\Delta & \longmapsto & \infty \end{array} \tag{1.2}$$

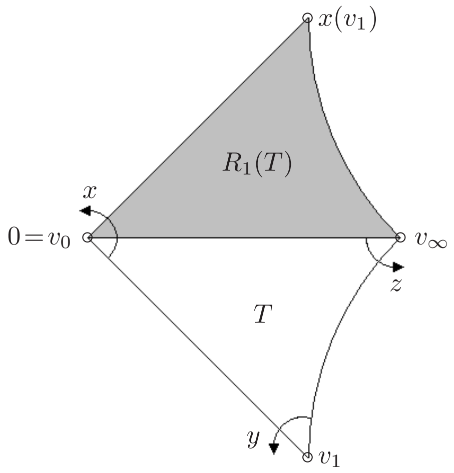


FIGURE 1.1. Generators x, y and z together with a fundamental domain of $\Delta(l,m,n)$ (depicted inside the unit disc model of the hyperbolic plane).

These groups are rigid among Fuchsian groups, in the sense that the quotient orbifold $\mathbb{H}/\Delta(l,m,n)$ does not admit non-trivial deformations. This rigidity together with Nielsen's theorem, that states that any group automorphism of a Fuchsian group Δ is induced by conjugation by a homeomorphism of the half-plane, implies that group automorphisms of a triangle group $\Delta(l,m,n)$ are induced by isometries of the hyperbolic plane. Actually, this result can be proved without using such advanced techniques.

LEMMA 1.1. *The automorphism group of a triangle group Δ is*

$$\mathrm{Aut}(\Delta) = \langle N(\Delta), \tau \rangle ,$$

where $N(\Delta)$, the normaliser of Δ in $\mathrm{PSL}(2,\mathbb{R})$, acts by conjugation and τ is determined by

$$\begin{array}{rccc} \tau: & \Delta & \longrightarrow & \Delta \\ & x & \longmapsto & x^{-1} \\ & y & \longmapsto & y^{-1} \\ & z & \longmapsto & yz^{-1}y^{-1} \end{array}$$

PROOF. Let us denote $X = \tau(x)$, $Y = \tau(y)$ and $Z = \tau(z)$. Being elliptic elements we can assume modulo conjugation in $N(\Delta)$ that they are conjugate to powers of x, y and z respectively. Let a, b and c be hyperbolic rotations through angles of $2\pi/l$, $2\pi/m$ and $2\pi/n$ around u, v and w, and let $X = a^k$, $Y = b^j$ and $Z = (a^k b^j)^{-1} = c^h$, $0 < k < l$, $0 < j < m$ and $0 < h < n$ (note that $(k,l) = (j,m) = (h,n) = 1$). Modulo conjugation with $\gamma \in \Delta$ we can suppose that u is the origin, and therefore $X = x^k$.

We will use the following formulae of hyperbolic geometry (see for example [10], chapter 7). The first one is the cosine rule for hyperbolic triangles. For any hyperbolic triangle with angles α, β and γ, the length A of the edge opposite to the vertex of angle α is given by

$$\cosh(A) = \frac{\cos(\gamma)\cos(\beta) + \cos(\alpha)}{\sin(\gamma)\sin(\beta)}\,.$$

The next ones relate the angles of rotation of elements of $\mathrm{PSL}(2,\mathbb{R})$ with their traces. Let ϕ and ψ be hyperbolic rotations of angle 2α and 2β, with $\alpha, \beta \in (0,\pi)$, around points u and v ($u \neq v$), and let us suppose furthermore that their sense of rotation is the same. Then

$$\begin{aligned} \mathrm{Tr}(\phi)^2 &= 4\cos^2(\alpha)\,, \\ |\mathrm{Tr}(\phi \circ \psi)| &= 2(\cosh\rho(u,v)\sin\alpha\sin\beta - \cos\alpha\cos\beta)\,. \end{aligned}$$

Applying these last two formulae to $(a^k b^j)^{-1} = c^h$, isolating the term $\cosh\rho(0,v)$ and considering the possible values of k, j and h we get the following inequality

$$\cosh\rho(0,v) = \frac{\cos\left(\frac{k\pi}{l}\right)\cos\left(\frac{j\pi}{m}\right) + \left|\cos\left(\frac{h\pi}{n}\right)\right|}{\sin\left(\frac{k\pi}{l}\right)\sin\left(\frac{j\pi}{m}\right)} \leq \frac{\cos\left(\frac{\pi}{l}\right)\cos\left(\frac{\pi}{m}\right) + \cos\left(\frac{\pi}{n}\right)}{\sin\left(\frac{\pi}{l}\right)\sin\left(\frac{\pi}{m}\right)}\,, \tag{1.3}$$

with equality if and only if $k \equiv \pm 1 \bmod l$, $j \equiv \pm 1 \bmod m$ and $h \equiv \pm 1 \bmod n$.

Now consider any triangle of the tessellation with a vertex in the origin, and call v' the vertex which is fixed by an element conjugate to z, which has angle π/m. By the cosine rule, the right-hand part of the inequality in (1.3) is exactly $\cosh\rho(0,v')$, and hence

$$\cosh\rho(0,v) \leq \cosh\rho(0,v') = \frac{\cos\left(\frac{\pi}{l}\right)\cos\left(\frac{\pi}{m}\right) + \cos\left(\frac{\pi}{n}\right)}{\sin\left(\frac{\pi}{l}\right)\sin\left(\frac{\pi}{m}\right)}$$

But it is clear that there are no vertices of the tessellation of angle π/m closer to the origin than v', and therefore $\cosh\rho(0,v) = \cosh\rho(0,v')$ and v is the vertex of a triangle with another vertex in the origin. As a consequence X, Y and Z are rotations of angles $\frac{2\pi}{p}$, $\frac{2\pi}{q}$ and $\frac{2\pi}{r}$ around the three vertices of a triangle.

On the other hand, the only possibilities for k and j are $k = j = 1$ or $k = l-1$, $j = m-1$. Therefore, either $X = x$ and $Y = x^\varepsilon y x^{-\varepsilon}$ or $X = x^{-1}$ and $Y = x^\varepsilon y^{-1} x^{-\varepsilon}$, for some integer ε. □

Corollary 1.1. *Any group isomorphism between triangle groups is induced by an isometry of the hyperbolic plane.*

Proof. The only thing left to prove is that the isomorphism τ just introduced is induced by an isometry of the hyperbolic plane. But clearly τ is induced by reflection on the geodesic passing through $v_0 = 0$ and v_1. □

This fact motivates the study of normalisers of triangle groups. It is a well-known fact (see [61]) that the normaliser $N(\Delta)$ in $\mathrm{PSL}(2,\mathbb{R})$ of a triangle group $\Delta \equiv \Delta(l,m,n)$ is a triangle group again, and that the quotient $N(\Delta)/\Delta$ is faithfully represented in the symmetric group $\mathfrak{S}_3$ via its action on the vertices $[v_0]$, $[v_1]$, $[v_\infty]$ of the orbifold $\mathbb{H}/\Delta$. Thus

$$N(\Delta)/\Delta \cong \begin{cases} \{1\}, & \text{if } l,\ m \text{ and } n \text{ are all distinct;} \\ \mathfrak{S}_2, & \text{if } l = m \neq n; \\ \mathfrak{S}_3, & \text{if } l = m = n. \end{cases} \tag{1.4}$$

where $\mathfrak{S}_k$ stands for the symmetric group on k elements.

In the second case, a representative for the non-trivial element $(1,2) \in \mathfrak{S}_2$ is the rotation $\lambda_4 \in N(\Delta)$ of order two around the midpoint of the segment joining v_0 and v_1 (see Figure 1.2). Conjugation by this element yields an order two automorphism of Δ which interchanges x and y and sends z to $x^{-1}zx$. We will denote it by $\tilde{\sigma}_4$.

In the case when $l = m = n$ we can choose the same representative λ_4 for the element $(1,2) \in \mathfrak{S}_3$, and the order three rotation λ_1 in the positive sense around the incentre of T (i.e. the point where the three angle bisectors meet, see [10] §7.14) for $(1,2,3) \in \mathfrak{S}_3$. Conjugation by the latter induces an automorphism $\tilde{\sigma}_1$ of Δ of order three which sends x to y and y to z (see Figure 1.2).

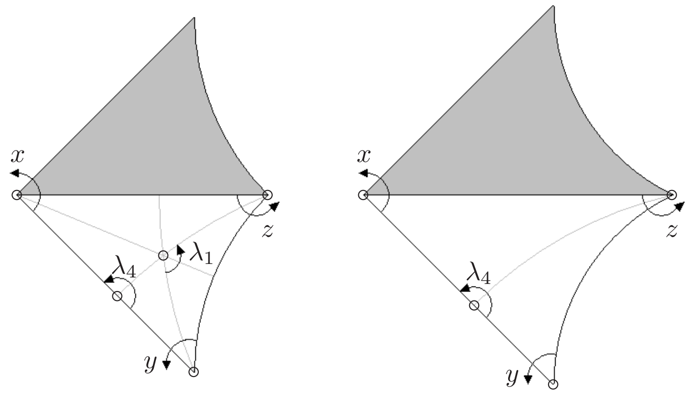


Figure 1.2. Generators of $\Delta(l,l,l)$ and $\Delta(l,l,n)$, and representatives of $(1,2), (1,2,3) \in \mathfrak{S}_3$.

In Table 1 a representative λ_i, $i = 0, \ldots, 5$, is chosen for each element of $\mathfrak{S}_3 \cong N(\Delta)/\Delta$, and for each automorphism $\tilde{\sigma}_i$ of Δ obtained by conjugation by λ_i, its action on the triple of generators x, y, z is indicated. The table describes the case in which $l = m = n$, but the other two cases are also contained in it, for obviously the case $l = m \neq n$ corresponds to the first and the fifth lines, and the case where l, m, n are all different corresponds to just the identity.

It is worth noting that in the case when $N(\Delta)/\Delta = \mathfrak{S}_2$ or $\{1\}$ the extension splits, i.e $N(\Delta) = \Delta \times (N(\Delta)/\Delta)$, but when $N(\Delta)/\Delta = \mathfrak{S}_3$ it does not, since no Fuchsian group can contain a noncyclic finite group. This means that the representatives of $N(\Delta)/\Delta$ cannot be chosen naturally to form a complement of Δ.

To summarize, $N(\Delta)$ can be written as

$$N(\Delta) \cong \begin{cases} \Delta, & \text{if } l,\ m \text{ and } n \text{ are all distinct;} \\ \langle \Delta, \lambda_4 \rangle, & \text{if } l = m \neq n; \\ \langle \Delta, \lambda_1, \lambda_4 \rangle, & \text{if } l = m = n. \end{cases} \tag{1.5}$$

Permutation	Representatives of $N(\Delta)/\Delta$	$\mathrm{Aut}(\Delta)$	Action on the generators of Δ
Id	$\lambda_0 = \mathrm{Id}$	$\tilde{\sigma}_0 \equiv \mathrm{Id}$	(x, y, z)
$(1,2,3)$	λ_1	$\tilde{\sigma}_1 : \gamma \mapsto \lambda_1 \gamma \lambda_1^{-1}$	(y, z, x)
$(1,3,2)$	$\lambda_2 = \lambda_1^2$	$\tilde{\sigma}_2 : \gamma \mapsto \lambda_2 \gamma \lambda_2^{-1}$	(z, x, y)
$(1,3)$	$\lambda_3 = \lambda_1 \lambda_4$	$\tilde{\sigma}_3 : \gamma \mapsto \lambda_3 \gamma \lambda_3^{-1}$	$(z, y, y^{-1}xy)$
$(1,2)$	λ_4	$\tilde{\sigma}_4 : \gamma \mapsto \lambda_4 \gamma \lambda_4^{-1}$	$(y, x, x^{-1}zx)$
$(2,3)$	$\lambda_5 = \lambda_1^2 \lambda_4$	$\tilde{\sigma}_5 : \gamma \mapsto \lambda_5 \gamma \lambda_5^{-1}$	$(x, z, z^{-1}yz)$

TABLE 1. Correspondence $N(\Delta)/\Delta \cong \mathfrak{S}_3$.

1.2. Triangle groups and dessins d'enfants

The importance of triangle groups in Grothendieck's theory of dessins d'enfants comes from the fact that any Belyi function β in a Riemann surface S can be represented as the natural projection $\mathbb{H}/\Lambda \longrightarrow \mathbb{H}/\Delta(l,m,n)$ from the quotient surface $\mathbb{H}/\Lambda \cong S$ to an orbifold $\mathbb{H}/\Delta(l,m,n)$ given by the inclusion $\Lambda < \Delta(l,m,n)$, where the signature of $\Delta(l,m,n)$ depends on the ramification orders of β ([**16, 68**]).

We have two important families of dessins. A dessin d'enfant $\mathcal{D}$ of type (l,m,n) (and its associated Belyi function) on a Riemann surface S is called *uniform* if all white vertices, black vertices and faces have degree l, m and n respectively. In the specific case where β is a uniform Belyi function of type (l,m,n), it corresponds to the inclusion of a torsion-free group K in the triangle group $\Delta(l,m,n)$. The group K is, of course, isomorphic to the fundamental group $\pi_1(S)$.

If, moreover, the automorphism group $\mathrm{Aut}(S)$ acts transitively on the edges of the dessin, we say that $\mathcal{D}$ is *regular*. A regular Belyi function corresponds to the normal inclusion of a uniformising group K of S in $\Delta(l,m,n)$, so that $\mathbb{H}/K \longrightarrow \mathbb{H}/\Delta(l,m,n)$ is a Galois covering with group $G \cong \Delta(l,m,n)/K$. Riemann surfaces which admit a regular Belyi function are called *quasiplatonic curves* (or triangle curves). In the next section we will make this connection between quasiplatonic curves and their covering groups G more explicit.

In these two cases one can study renormalisations of the dessin. Suppose that $\mathcal{D}$ is a uniform dessin on a Riemann surface S associated to a Belyi function β, and suppose that some of the orders of its type (l,m,n) are repeated. Then one can construct other dessins on the same surface by renormalisation in the following way. Consider an automorphism $F : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ of the Riemann sphere which permutes the ramification values of β of the same order, i.e. F permutes, for instance, 0 and 1 if $l = m \neq n$, and F permutes 0, 1 and ∞ if $l = m = n$. In this way the map $\beta_F = F \circ \beta$ is a Belyi function again, and the corresponding dessin $\mathcal{D}_F$ is called a renormalised dessin of $\mathcal{D}$.

Now, if the original Belyi function was given by an inclusion $K < \Delta$, the renormalised function β_F is induced by an element of $N(\Delta)$ in the following way: there exists $\alpha \in N(\Delta)$ whose action on v_0, v_1 and v_∞ coincides with the action of F on 0, 1 and ∞, and then β_F is given by the inclusion $\alpha K \alpha^{-1} < \Delta$, since one has

$$\beta_F : \mathbb{H}/K \xrightarrow{\ \beta\ } \mathbb{H}/\Delta \xrightarrow{\ \alpha\ } \mathbb{H}/\Delta$$

In the case when α additionally belongs to $N(K)$, there exists an isomorphism $\phi \in \mathrm{Aut}(S)$ such that $\beta \circ \phi = \beta_F$, and $\mathcal{D}_F$ and $\mathcal{D}$ are isomorphic.

1.3. Triangle coverings

Now let G be a finite group, S a compact Riemann surface and $\mathrm{Aut}(S)$ its automorphism group. By a *triangle $G-$covering* (or a $G-$orbifold of genus zero) of type (l, m, n) we will understand a quasiplatonic curve S together with a regular Belyi function $f : S \longrightarrow \mathbb{P}^1$, ramified over 0, 1 and ∞ with orders l, m and n respectively, such that the group of deck transformations $\mathrm{Aut}(S, f)$ is isomorphic to the group G. Under these assumptions there is a monomorphism $\mathfrak{i} : G \longrightarrow \mathrm{Aut}(S)$ where $\mathfrak{i}(G)$ agrees with the covering group $\mathrm{Aut}(S, f)$ consisting of the elements $\tau \in \mathrm{Aut}(S)$ such that $f \circ \tau = f$, so that the Belyi function f agrees with the quotient map $S \longrightarrow S/\mathfrak{i}(G)$. Note that $\mathfrak{i}$ is only determined up to composition with an element of $\mathrm{Aut}(G)$. We will write (S, f) for such a $G-$covering, and we will always suppose that it is hyperbolic, i.e. that the genus of S is $g(S) \geq 2$.

Given (S_1, f_1) and (S_2, f_2) we say that an isomorphism $\tau : S_2 \longrightarrow S_1$ is a *strict isomorphism* of $G-$coverings if $f_2 = f_1 \circ \tau$, and we call it a *twisted isomorphism* if $f_2 = F \circ f_1 \circ \tau$ for some automorphism F of $\mathbb{P}^1$. These two concepts can be better visualized by means of the following two commutative diagrams

$$\begin{array}{ccc} S_1 & \xleftarrow{\ \tau\ } & S_2 \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_2} \\ \mathbb{P}^1 & \xrightarrow{\ \mathrm{Id}\ } & \mathbb{P}^1 \end{array} \qquad\qquad \begin{array}{ccc} S_1 & \xleftarrow{\ \tau\ } & S_2 \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_2} \\ \mathbb{P}^1 & \xrightarrow{\ F\ } & \mathbb{P}^1 \end{array}$$

Triangle $G-$coverings can be studied in a purely group theoretical way. We say that a triple (a, b, c) of elements generating G is a *hyperbolic triple of generators* of G of type (l, m, n) if the following conditions hold:

(i) $abc = 1$;
(ii) $\mathrm{ord}(a) = l$, $\mathrm{ord}(b) = m$ and $\mathrm{ord}(c) = n$;
(iii) $\frac{1}{l} + \frac{1}{m} + \frac{1}{n} < 1$.

To such a hyperbolic triple of generators we can associate a triangle $G-$covering of type (l, m, n) in the following way. The kernel K of the epimorphism

$$\begin{array}{rccc} \rho : & \Delta(l,m,n) & \longrightarrow & G \\ & x & \longmapsto & a \\ & y & \longmapsto & b \\ & z & \longmapsto & c \end{array} \tag{1.6}$$

is a torsion-free Fuchsian group so that $S = \mathbb{H}/K$ is a compact Riemann surface which carries a monomorphism $\mathfrak{i} : G \longrightarrow \mathrm{Aut}(S)$ given by the rule

$$\mathfrak{i}(g)([\mathrm{w}]_K) = [\delta(\mathrm{w})]_K, \text{ for any choice of } \delta \in \Delta \text{ such that } \rho(\delta) = g\,.$$

It follows that the natural projection $\pi : \mathbb{H}/K \longrightarrow \mathbb{H}/\Delta$ induces a triangle $G-$covering (S, f) of type (l, m, n) defined by the commutative diagram

$$\begin{array}{ccc} S = \mathbb{H}/K & & \\ \downarrow & \searrow^{f} & \\ \mathbb{H}/\Delta & \xrightarrow{\ \Phi\ } & \mathbb{P}^1 \end{array} \tag{1.7}$$

where Φ is as in (1.2).

Such a covering is hyperbolic precisely because the orders l, m and n satisfy condition (iii) above, as by the Riemann–Hurwitz formula the genus $g(S)$ of S is given by

$$2g(S) - 2 = |G| \left(1 - \left(\frac{1}{l} + \frac{1}{m} + \frac{1}{n}\right)\right) . \tag{1.8}$$

Consider the action of $\mathrm{Aut}(G)$ on triples given by $\psi(a,b,c) := (\psi(a), \psi(b), \psi(c))$ for $\psi \in \mathrm{Aut}(G)$. Clearly the triples (a,b,c) and $\psi(a,b,c)$ give rise to the same $G-$cover.

Conversely a hyperbolic triangle $G-$covering (S,f) of type (l,m,n) determines a triple of generators of G, defined up to an element of $\mathrm{Aut}(G)$, in the following manner. Uniformisation theory tells us that there is a torsion-free Fuchsian group K_1 uniformising S, whose normaliser $N(K_1)$ contains $\Delta = \Delta(l,m,n)$, and an isomorphism of coverings of the form

$$\begin{array}{ccc} \mathbb{H}/K_1 & \xrightarrow{\widetilde{u}} & S \\ \downarrow & & \downarrow f \\ \mathbb{H}/\Delta & \xrightarrow{u} & \mathbb{P}^1 \end{array}$$

If the orders l, m and n are all distinct then necessarily u agrees with the isomorphism Φ defined in (1.2). Otherwise note that any element of $N(\Delta)$ induces an automorphism of $\mathbb{H}/\Delta$ which permutes the points $[v_0]_\Delta$, $[v_1]_\Delta$ and $[v_\infty]_\Delta$ with equal orders. Therefore there is an element $\alpha \in N(\Delta)$ producing the following commutative diagram

$$\begin{array}{ccccc} \mathbb{H}/\alpha^{-1}K_1\alpha & \xrightarrow{\alpha} & \mathbb{H}/K_1 & \xrightarrow{\widetilde{u}} & S \\ \downarrow & & \downarrow & & \downarrow f \\ \mathbb{H}/\Delta & \xrightarrow{\alpha} & \mathbb{H}/\Delta & \xrightarrow{u} & \mathbb{P}^1 \end{array} \tag{1.9}$$

where $u \circ \alpha$ equals Φ. Thus, replacing $\widetilde{\Phi}$ with $\widetilde{u} \circ \alpha$ and $\alpha^{-1}K_1\alpha$ with K, one always has a diagram of the form

$$\begin{array}{ccc} \mathbb{H}/K & \xrightarrow{\widetilde{\Phi}} & S \\ \downarrow & & \downarrow f \\ \mathbb{H}/\Delta & \xrightarrow{\Phi} & \mathbb{P}^1 \end{array} \tag{1.10}$$

This yields an epimorphism $\rho : \Delta \longrightarrow G$ (which is defined only up to an automorphism of G, just as the monomorphism $\mathfrak{i}$ is) determined by the identity

$$\widetilde{\Phi}([\gamma(\mathrm{w})]) = \mathfrak{i}\,(\rho(\gamma))\,\widetilde{\Phi}([\mathrm{w}]) \tag{1.11}$$

for all $\gamma \in \Delta$, and hence a hyperbolic triple of generators

$$(a,b,c) := (\rho(x), \rho(y), \rho(z))\,.$$

1.3.1. Strict equivalence of triangle $G-$coverings. If in the above discussion, we start with a triangle $G-$covering (S', f') strictly isomorphic to (S,f) by means of a strict isomorphism $\tau : (S', f') \longrightarrow (S,f)$ and choose corresponding Fuchsian group representations we get a diagram as follows

$$\begin{array}{ccccccc} S & = & \mathbb{H}/K & \xleftarrow{\tau} & \mathbb{H}/K' & = & S' \\ & & f\downarrow & & \downarrow f' & & \\ \mathbb{P}^1 & = & \mathbb{H}/\Delta & \xrightarrow{\mathrm{Id}} & \mathbb{H}/\Delta & = & \mathbb{P}^1 \end{array}$$

We observe that, in order for this diagram to be commutative, the isomorphism $\tau^{-1} : \mathbb{H}/K \longrightarrow \mathbb{H}/K'$ must be induced by an element $\delta \in \Delta$. We see that the isomorphism $\widetilde{\Phi'} : \mathbb{H}/K' \longrightarrow S'$ defining the diagram analogous to (1.10) for the pair (S', f') is given by $\widetilde{\Phi'} = \tau^{-1} \circ \widetilde{\Phi} \circ \delta^{-1}$. Plugging this expression into the corresponding formula (1.11), which now reads $\widetilde{\Phi'}([\gamma(\mathrm{w})]) = \mathfrak{i}'(\rho'(\gamma))\, \widetilde{\Phi'}([\mathrm{w}])$, we get the identity

$$\tau^{-1} \circ \mathfrak{i}\left(\rho(\delta^{-1}\gamma)\right) = \mathfrak{i}'\left(\rho'(\gamma)\right) \circ \tau^{-1} \circ \mathfrak{i}\left(\rho(\delta^{-1})\right) .$$

It follows that $\rho' = \psi \circ \rho$, where $\psi \in \mathrm{Aut}(G)$ is defined by $\psi(g) = (\mathfrak{i}')^{-1}(g_0 \cdot \mathfrak{i}(g) \cdot g_0^{-1})$ with $g_0 = \tau^{-1} \circ \mathfrak{i}\left(\rho(\delta^{-1})\right)$. As a consequence $(a', b', c') = \psi(a, b, c)$ and we have the following proposition.

Proposition 1.1. *There is a bijection*

$$\left\{ \begin{array}{c} \textit{Strict isomorphism classes} \\ \textit{of triangle } G\textit{-covers } (S, f) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \textit{Hyp. triples of generators} \\ \textit{of } G \textit{ modulo } \mathrm{Aut}(G) \end{array} \right\}$$

1.3.2. Twisted equivalence of triangle G−coverings. In order to prove the analogous result of Proposition 1.1 for twisted coverings we need to identify triples of generators modulo the action of a larger group.

Given a finite group G, we introduce for convenience the following bijections of the set $\mathbb{T}(G; l, m, n)$ of hyperbolic triples of generators of G of a given type (l, m, n). They are defined in the following way (cf. [5])

$$\begin{array}{ll} \sigma_0(a,b,c) = (a,b,c)\,, & \sigma_3(a,b,c) = (c,b,b^{-1}ab)\,, \\ \sigma_1(a,b,c) = (b,c,a)\,, & \sigma_4(a,b,c) = (b,a,a^{-1}ca)\,, \\ \sigma_2(a,b,c) = (c,a,b)\,, & \sigma_5(a,b,c) = (a,c,c^{-1}bc)\,. \end{array} \tag{1.12}$$

Note that they are defined so as to satisfy

$$\begin{aligned} \sigma_i(\rho(x), \rho(y), \rho(z)) &= \left(\rho\left(\lambda_i x \lambda_i^{-1}\right), \rho\left(\lambda_i y \lambda_i^{-1}\right), \rho\left(\lambda_i z \lambda_i^{-1}\right)\right) = \\ &= \left(\rho\left(\tilde{\sigma}_i(x)\right), \rho\left(\tilde{\sigma}_i(y)\right), \rho\left(\tilde{\sigma}_i(z)\right)\right) , \end{aligned} \tag{1.13}$$

where $\rho : \Delta(l, m, n) \longrightarrow G$ is the epimorphism associated in (1.6) to each triple of generators.

In order to understand the relationship between triples of generators of G and twisted isomorphism classes of triangle G−coverings, we will need to consider the following group of bijections of $\mathbb{T}(G; l, m, n)$

$$\mathrm{A}(G; l, m, n) = \begin{cases} \mathrm{Aut}(G), & \text{if } l,\ m,\ n \text{ are all distinct;} \\ \langle\, \mathrm{Aut}(G), \sigma_4 \,\rangle, & \text{if } l = m \neq n; \\ \langle\, \mathrm{Aut}(G), \sigma_1, \ldots, \sigma_5 \,\rangle, & \text{if } l = m = n. \end{cases}$$

The action of the composition of two elements σ_i and σ_j on a triple (a, b, c) follows the following table

	$\boldsymbol{\sigma_0}$	$\boldsymbol{\sigma_1}$	$\boldsymbol{\sigma_2}$	$\boldsymbol{\sigma_3}$	$\boldsymbol{\sigma_4}$	$\boldsymbol{\sigma_5}$
$\boldsymbol{\sigma_0}$	σ_0	σ_1	σ_2	σ_3	σ_4	σ_5
$\boldsymbol{\sigma_1}$	σ_1	σ_2	σ_0	$\gamma_{b^{-1}} \circ \sigma_4$	$\gamma_{a^{-1}} \circ \sigma_5$	$\gamma_{c^{-1}} \circ \sigma_3$
$\boldsymbol{\sigma_2}$	σ_2	σ_0	σ_1	$\gamma_c \circ \sigma_5$	$\gamma_b \circ \sigma_3$	$\gamma_a \circ \sigma_4$
$\boldsymbol{\sigma_3}$	σ_3	σ_5	σ_4	$\gamma_{b^{-1}} \circ \sigma_0$	$\gamma_{a^{-1}} \circ \sigma_2$	$\gamma_{c^{-1}} \circ \sigma_1$
$\boldsymbol{\sigma_4}$	σ_4	σ_3	σ_5	σ_1	σ_0	σ_2
$\boldsymbol{\sigma_5}$	σ_5	σ_4	σ_3	$\gamma_c \circ \sigma_2$	$\gamma_b \circ \sigma_1$	$\gamma_a \circ \sigma_0$

where the product $\sigma_i \cdot \sigma_j$ is to be found in the intersection of the $i-$th row and the $j-$th column, and γ_g stands for conjugation by an element $g \in G$. Using this table, one can easily check that G is normal in $\mathrm{A}(G;l,m,n)$.

As a consequence the action of any element $\mu \in \mathrm{A}(G;l,m,n)$ on a specific triple (a,b,c) can be written as $\mu = \psi \circ \sigma_i$ for some σ_i, $i = 0,\dots,5$, where ψ is an automorphism of G. We note that in general ψ depends on the triple (a,b,c).

Given an element $\delta \in \mathrm{PSL}(2,\mathbb{R})$, we will write φ_δ for conjugation by δ.

LEMMA 1.2. *The following two statements are equivalent:*

(i) $(a,b,c) \equiv (a',b',c') \bmod \mathrm{A}(G;l,m,n)$*;*
(ii) *there exist* $\psi \in \mathrm{Aut}(G)$ *and* $\delta \in N(\Delta)$ *such that* $\rho' = \psi \circ \rho \circ \varphi_\delta$*.*

PROOF. Let us suppose that $(a,b,c) \equiv (a',b',c') \bmod \mathrm{A}(G;l,m,n)$. By the comments above there exists a transformation σ_i such that $(a',b',c') = \psi(\sigma_i(a,b,c))$. Therefore, using (1.13), we have

$$(\rho'(x),\rho'(y),\rho'(z)) = (a',b',c') = \psi(\rho(\tilde\sigma_i(x)),\rho(\tilde\sigma_i(y)),\rho(\tilde\sigma_i(z)))\,.$$

For the converse, note that by (1.5) every $\delta \in N(\Delta)$ is of the form $\delta = \eta\lambda_i$, for some λ_i, $i = 0,\dots,5$ and $\eta \in \Delta$.

Therefore, we can write

$$(a',b',c') = \psi\,(\rho(\varphi_\delta(x)),\rho(\varphi_\delta(y)),\rho(\varphi_\delta(z))) = \psi(g(\sigma_i(a,b,c))g^{-1})\,,$$

where $g = \rho(\eta)$. □

For later use we record the following remark.

REMARK 1.1. If instead of the group $\mathrm{A}(G;l,m,n)$ we restrict ourselves to the subgroup

$$\mathrm{I}(G;l,m,n) = \begin{cases} G, & \text{if } l,\ m,\ n \text{ are all distinct;} \\ \langle G,\sigma_4\rangle, & \text{if } l = m \neq n; \\ \langle G,\sigma_1,\sigma_4\rangle, & \text{if } l = m = n. \end{cases}$$

where G acts on $\mathbb{T}(G;l,m,n)$ by conjugation, then the corresponding result in Lemma 1.2 will be that $(a,b,c) \equiv (a',b',c') \bmod \mathrm{I}(G;l,m,n)$ if and only if $\rho' = \rho \circ \varphi_\delta$ for some $\delta \in N(\Delta)$.

More precisely, if $(a',b',c') = g \cdot (\sigma_i(a,b,c)) \cdot g^{-1}$, then the element $\delta \in N(\Delta)$ can be taken to be $\delta = \eta\lambda_i$, for any $\eta \in \Delta$ such that $g = \rho(\eta)$.

We can now prove the analogue of Proposition 1.1 for the twisted case, namely

PROPOSITION 1.2. *There is a bijection*

$$\left\{ \begin{array}{c} \textit{Twisted isomorphism classes} \\ \textit{of triangle } G\textit{−covers } (S,f) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \textit{Hyp. triples of generators} \\ \textit{of } G \textit{ modulo } \mathrm{A}(G;l,m,n) \end{array} \right\}$$

PROOF. Let (a,b,c), (a',b',c') be two triples of hyperbolic generators of G determining two epimorphisms ρ and ρ', and hence two triangle $G-$coverings as in (1.7). If $(a',b',c') \equiv (a,b,c) \bmod \mathrm{A}(G;l,m,n)$ then by Lemma 1.2 one has the equality $K' := \ker\rho' = \delta K \delta^{-1}$ and a commutative diagram as follows

$$\begin{array}{ccccc} S = & \mathbb{H}/K & \xrightarrow{\ \delta\ } & \mathbb{H}/K' & = S' \\ & {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle f'} & \\ & \mathbb{P}^1 & \xrightarrow{\ F\ } & \mathbb{P}^1 & \end{array}$$

where $F = \Phi \circ \overline{\delta} \circ \Phi^{-1}$ and $\overline{\delta}$ is the automorphism of $\mathbb{H}/\Delta$ induced by δ. Therefore, in this case, the corresponding coverings (S, f) and (S', f') are twisted isomorphic.

Conversely, if we start with a twisted isomorphism of coverings τ between (S, f) and (S', f'), then there is a commutative diagram of the form

$$\begin{array}{ccccc} S & \xrightarrow{\mathrm{Id}} & S & \xrightarrow{\tau} & S' \\ f\downarrow & & \downarrow f_1 & & \downarrow f' \\ \mathbb{P}^1 & \xrightarrow{F} & \mathbb{P}^1 & \xrightarrow{\mathrm{Id}} & \mathbb{P}^1 \end{array}$$

where $(S, f_1) := (S, F \circ f)$ for a suitable Möbius transformation F. Since (S, f_1) and (S', f') are strictly isomorphic, there is an automorphism $\psi \in \mathrm{Aut}(G)$ such that their corresponding epimorphisms ρ_1 and ρ' are related by $\rho_1 = \psi \circ \rho'$. Now, as explained in the previous sections (see (1.9) and (1.10)), from the Fuchsian group point of view the coverings (S, f) and (S, f_1) correspond to diagrams

$$\begin{array}{ccc} \mathbb{H}/K & \xrightarrow{\widetilde{\Phi}} & S \\ \downarrow & & \downarrow f \\ \mathbb{H}/\Delta & \xrightarrow{\Phi} & \mathbb{P}^1 \end{array} \qquad \begin{array}{ccc} \mathbb{H}/\delta^{-1}K\delta & \xrightarrow{\widetilde{\Phi}_1} & S \\ \downarrow & & \downarrow F\circ f \\ \mathbb{H}/\Delta & \xrightarrow{\Phi} & \mathbb{P}^1 \end{array}$$

where $\widetilde{\Phi}_1 = \widetilde{\Phi} \circ \delta$ and $\delta \in N(\Delta)$ induces the automorphism $\overline{\delta} : \mathbb{H}/\Delta \longrightarrow \mathbb{H}/\Delta$ such that $F \circ \Phi \circ \overline{\delta} = \Phi$. As a consequence the epimorphism ρ_1 corresponding to $(S, F \circ f)$ is defined by the equality

$$\widetilde{\Phi}_1([\gamma(\mathrm{w})]) = \mathfrak{i}\,(\rho_1(\gamma))\,\widetilde{\Phi}_1([\mathrm{w}])\,,$$

and therefore $\rho_1(\gamma) = \rho(\delta\gamma\delta^{-1})$.

By Lemma 1.2, since $\rho(\gamma) = \rho_1(\delta^{-1}\gamma\delta) = \psi \circ \rho'(\delta^{-1}\gamma\delta)$, we finally have that $(a, b, c) \equiv (a', b', c') \bmod \mathrm{A}(G; l, m, n)$. □

1.4. Galois conjugation of triangle curves

It is known (see [37]) that both $G-$covers and its automorphism groups can be simultaneously defined over $\overline{\mathbb{Q}}$. This permits an action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on equivalence classes of $G-$coverings (S, f). For an element $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ one simply defines $(S, f, \mathfrak{i})^\sigma := (S^\sigma, f^\sigma, \mathfrak{i}^\sigma)$, where $f^\sigma : S^\sigma \longrightarrow \mathbb{P}^1$ is obtained by applying σ to the coefficients defining the covering $f : S \longrightarrow \mathbb{P}^1$ and $\mathfrak{i}^\sigma : G \longrightarrow \mathrm{Aut}(S^\sigma)$ is defined by $\mathfrak{i}^\sigma(h) = (\mathfrak{i}(h))^\sigma$.

This rather canonical action of the absolute Galois group on $G-$covers turns out to be very mysterious at the level of triples of generators, their equivalent counterparts in Proposition 1.1. One way to gain some insight on it is by relating the rotation numbers of these generators at certain points of S to their rotation numbers at the corresponding points of S^σ. Let us stress here that in [63] M. Streit used rotation numbers to study the action of the Galois group on quasiplatonic curves uniformised by normal subgroups of $\Delta(2, 3, n)$ with quotient group isomorphic to $\mathrm{PSL}(2, p)$. Here we sum up Streit's method in a more general context.

Proposition 1.3. *Let (a, b, c) be a hyperbolic triple of generators of G of type (l, m, n) defining a $G-$covering (S, f). Then for any $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ the $G-$covering (S^σ, f^σ) corresponds to a hyperbolic triple of generators $(a_\sigma, b_\sigma, c_\sigma)$ of G of the form*

$$a_\sigma = h_a a^\alpha h_a^{-1}\,, \qquad b_\sigma = h_b b^\beta h_b^{-1}\,, \qquad c_\sigma = h_c c^\gamma h_c^{-1}\,,$$

where $\sigma(\zeta_l^\alpha) = \zeta_l$, $\sigma(\zeta_m^\beta) = \zeta_m$ *and* $\sigma(\zeta_n^\gamma) = \zeta_n$ *and* $h_a, h_b, h_c \in G$.

Proof. First of all note that, since by formula (1.11) the action of the element $a \in G$ (resp. b, resp. c) at the point $P_0 = \widetilde{\Phi}([v_0]_K)$ (resp. $P_1 = \widetilde{\Phi}([v_1]_K)$, resp. $P_\infty = \widetilde{\Phi}([v_\infty]_K)$) is locally described by the action of the element $x \in \Delta$ (resp. y, resp. z), we may conclude that the element a (resp. b, resp. c) possesses one fixed point in the fibre of 0 (resp. of 1, resp. of ∞) with rotation number ζ_l (resp. ζ_m, resp. ζ_n).

Suppose now that $(a_\sigma, b_\sigma, c_\sigma)$ is a hyperbolic triple of generators of G defining the $G-$covering $(S^\sigma, f^\sigma, \mathfrak{i}^\sigma)$. This means that if K_σ is the kernel of the epimorphism

$$\begin{array}{rccc} \rho_\sigma : & \Delta(l,m,n) & \longrightarrow & G \\ & x & \longmapsto & a_\sigma \\ & y & \longmapsto & b_\sigma \\ & z & \longmapsto & c_\sigma \end{array}$$

there is a commutative diagram

$$\begin{array}{ccc} \mathbb{H}/K_\sigma & \xrightarrow{\widetilde{\Phi}} & S^\sigma \\ \big\downarrow & & \big\downarrow f^\sigma \\ \mathbb{H}/\Delta & \xrightarrow{\Phi} & \mathbb{P}^1 \end{array}$$

such that a_σ (resp. b_σ, resp. c_σ) fixes a point $P_{0,\sigma} \in (f^\sigma)^{-1}(0)$ (resp. $P_{1,\sigma} \in (f^\sigma)^{-1}(1)$, resp. $P_{\infty,\sigma} \in (f^\sigma)^{-1}(\infty)$) with rotation angle ζ_l (resp. ζ_m, resp. ζ_n).

On the other hand, since a fixes the point $P_0 \in f^{-1}(0)$ with rotation number ζ_l then, by definition of the action of G on S^σ, a fixes the point $P_0^\sigma \in (f^\sigma)^{-1}(0)$ with rotation number $\sigma(\zeta_l)$. Since $P_{0,\sigma}$ and P_0^σ belong to the same fibre $(f^\sigma)^{-1}(0)$, there must be an element $h_a^{-1} \in G$ such that $\mathfrak{i}^\sigma(h_a^{-1})(P_{0,\sigma}) = P_0^\sigma$. Therefore $h_a a h_a^{-1}$ fixes the point $P_{0,\sigma}$ with rotation angle $\sigma(\zeta_l)$ and so does $h_a a^\alpha h_a^{-1}$, with rotation angle $\sigma(\zeta_l^\alpha) = \zeta_l$. As a consequence $a_\sigma = h_a a^\alpha h_a^{-1}$ and, proceeding in the same way with the other two generators, one gets

$$\begin{aligned} a_\sigma &= h_a a^\alpha h_a^{-1}\,, \\ b_\sigma &= h_b b^\beta h_b^{-1}\,, \\ c_\sigma &= h_c c^\gamma h_c^{-1}\,. \end{aligned}$$

□

Remark 1.2. (i) Note that through conjugation by an element of G, more precisely h_c^{-1}, we can always normalise the second triple so that for instance, $c_\sigma = c^{\gamma'}$.

(ii) Let $\sigma(\zeta_l) = \zeta_l^{\alpha'}$, $\sigma(\zeta_m) = \zeta_m^{\beta'}$ and $\sigma(\zeta_n) = \zeta_n^{\gamma'}$. Note that $\alpha \cdot \alpha' \equiv 1 \bmod l$, $\beta \cdot \beta' \equiv 1 \bmod m$ and $\gamma \cdot \gamma' \equiv 1 \bmod n$. These exponents $\alpha', \beta', \gamma' \in \mathbb{N}$ can be chosen to be equal, for if r is the least common multiple of the integers l, m, n and $\sigma(\zeta_r) = \zeta_r^\delta$ then one also has $\sigma(\zeta_l) = \zeta_l^\delta, \sigma(\zeta_m) = \zeta_m^\delta$ and $\sigma(\zeta_n) = \zeta_n^\delta$.

In the special case where σ is complex conjugation there is a precise formula for the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on triples

Proposition 1.4. *Let* (a, b, c) *be a hyperbolic triple of generators of* G *defining a* $G-$*covering* (S, f). *Then the complex conjugate* $G-$*covering* $(\overline{S}, \overline{f})$ *is defined by the triple* $(a^{-1}, ab^{-1}a^{-1}, c^{-1})$.

PROOF. We will work here with the unit disc $\mathbb{D}$ instead of the upper half-plane. We observe that if

$$\begin{array}{ccc} \mathbb{D}/K & \stackrel{\widetilde{\Phi}}{\longrightarrow} & S \\ \downarrow & & \downarrow f \\ \mathbb{D}/\Delta & \stackrel{\Phi}{\longrightarrow} & \mathbb{P}^1 \end{array}$$

is the commutative diagram (1.7) defining (S, f) then the covering $(\overline{S}, \overline{f})$ is defined by the diagram

$$\begin{array}{ccc} \mathbb{D}/\overline{K} & \stackrel{\widetilde{\Phi}_1}{\longrightarrow} & \overline{S} \\ \downarrow & & \downarrow \overline{f} \\ \mathbb{D}/\Delta & \stackrel{\Phi}{\longrightarrow} & \mathbb{P}^1 \end{array}$$

where for a subgroup H of $\mathrm{Aut}(\mathbb{D})$ we put $\overline{H} = \{\overline{h} : h \in H\}$ and $\widetilde{\Phi}_1(\mathrm{w}) = \overline{\widetilde{\Phi}}(\overline{\mathrm{w}})$.

Note that the function $\Phi_1(\mathrm{w}) = \overline{\Phi(\overline{\mathrm{w}})} = \overline{\Phi}(\overline{\mathrm{w}})$ induces the same isomorphism $\mathbb{D}/\Delta \simeq \mathbb{P}^1$ as Φ. Moreover, since $x(\mathrm{w}) = \zeta_l \cdot \mathrm{w}$ and z is conjugate to $\mathrm{w} \mapsto \zeta_n \cdot \mathrm{w}$ by means of a real Möbius transformation (see Figure 1.1) we see that $\overline{x} = x^{-1}$ and $\overline{z} = z^{-1}$. It follows that $\overline{\Delta} = \Delta$ and that the epimorphisms

$$\begin{array}{rccc} \rho: & \Delta(l,m,n) & \longrightarrow & G \\ & x & \longmapsto & a \\ & y & \longmapsto & b \\ & z & \longmapsto & c \end{array} \qquad \begin{array}{rccc} \overline{\rho}: & \Delta(l,m,n) & \longrightarrow & G \\ & x & \longmapsto & a^{-1} \\ & y & \longmapsto & ab^{-1}a^{-1} \\ & z & \longmapsto & c^{-1} \end{array}$$

are related by $\overline{\rho}(\gamma) = \rho(\overline{\gamma})$. We see that $\overline{K} = \ker(\overline{\rho})$ and the hyperbolic triple $(a^{-1}, ab^{-1}a^{-1}, c^{-1})$ defines the $G-$covering $(\overline{S}, \overline{f})$. □

Triangle curves and $G-$coverings are known to be defined over their fields of moduli ([68]). Proposition 1.3 immediately implies the following

COROLLARY 1.2. *Abelian $G-$coverings are defined over $\mathbb{Q}$.*

PROOF. Proposition 1.3 together with the second part of Remark 1.2 imply that if (a, b, c) is the triple defining an abelian $G-$covering (S, f) then, for any $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, the triple defining the covering (S^σ, f^σ) is of the form (a^k, b^k, c^k). Now these two triples differ by the automorphism ψ of G defined by $\psi(u) = u^k$. Hence, for any $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ the $G-$coverings (S, f) and (S^σ, f^σ) are equivalent. □

Alternative proofs of this fact have been found by R. Hidalgo ([45]) and B. Mühlbauer (forthcoming PhD thesis). See also the article by I. Bauer and F. Catanese [4].

1.5. Triangle curves with covering group G=PSL(2,p)

In this section we find hyperbolic triangle curves with covering group $G = \mathrm{PSL}(2, p) = \mathrm{SL}(2, p)/\{\pm \mathrm{Id}\}$. These will be used later, in section 3.4, in the construction of our Beauville surfaces.

Recall that if $p > 2$ is a prime, G is a group of order $p(p-1)(p+1)/2$, and observe that this expression already shows that it always has elements of orders 2, 3 and p. Conjugacy classes of elements and subgroups of $\mathrm{PSL}(2, p)$ are very well known. They can be found in almost any introductory book on linear groups (see for example [46] or [27] for an exhaustive exposition).

Throughout this section we will repeatedly use the following known result, which can be found for instance in [27], §5.2. If $p \geq 5$ is a prime, then the conjugacy class of an element of $\mathrm{PSL}(2,p)$ is determined by its trace, except for elements of order p which lie in two different classes and always have trace ± 2.

Now by the results of section 1.3, the study of $G-$coverings is equivalent to the study of triples of generators of $G = \mathrm{PSL}(2,p)$. These were studied by Macbeath in [50]. In order to present the results we need, we consider for any triple $(\alpha,\beta,\gamma) \in \mathbb{F}_p^*$ the set $E(\alpha,\beta,\gamma)$ that consists of all triples of elements (A,B,C) of $\mathrm{SL}(2,p)$ with traces α, β and γ respectively, such that their product is the identity. Consequently we write $\overline{E}(\alpha,\beta,\gamma)$ for the image of $E(\alpha,\beta,\gamma)$ in $\mathrm{PSL}(2,p)$.

A triple (α,β,γ) is called singular if

$$\alpha^2 + \beta^2 + \gamma^2 - \alpha\beta\gamma - 4 = 0$$

and exceptional if the orders of the elements in the triples of $\overline{E}(\alpha,\beta,\gamma)$ are one of the following:

$$(2,2,n), (2,3,3), (3,3,3), (3,4,4), (2,3,4),$$
$$(2,5,5), (5,5,5), (3,3,5), (3,5,5), (2,3,5).$$

Then Theorems 2 and 3 in [50] can be summarized as follows.

Theorem 1.1 (Macbeath). *A triple in $E(\alpha,\beta,\gamma)$ generates the whole group $\mathrm{PSL}(2,p)$ if and only if (α,β,γ) is neither singular nor exceptional. In this case:*

(i) *there are two conjugacy classes of triples in $E(\alpha,\beta,\gamma)$ modulo $\mathrm{SL}(2,p)$;*
(ii) *there is one conjugacy class of triples in $E(\alpha,\beta,\gamma)$ modulo $\mathrm{Aut}(\mathrm{SL}(2,p))$.*

To count the effective number of corresponding triples in $\mathrm{PSL}(2,p)$ we will use the following obvious observation.

Lemma 1.3. *Let (α,β,γ) and $E(\alpha,\beta,\gamma)$ be as above. Then in $\mathrm{PSL}(2,p)$*

$$\overline{E}(\alpha,\beta,\gamma) = \overline{E}(-\alpha,-\beta,\gamma) = \overline{E}(-\alpha,\beta,-\gamma) = \overline{E}(\alpha,-\beta,-\gamma)\,.$$

Proof. If we write (A,B,C) for a triple in $E(\alpha,\beta,\gamma)$, then clearly

$$(A,B,C) \in E(\alpha,\beta,\gamma) \iff (-A,-B,C) \in E(-\alpha,-\beta,\gamma) \iff$$
$$\iff (-A,B,-C) \in E(-\alpha,\beta,-\gamma) \iff (A,-B,-C) \in E(\alpha,-\beta,-\gamma)\,,$$

and these four triples project in $\mathrm{PSL}(2,p)$ to the same element. ☐

1.5.1. Type $(\mathbf{2},\mathbf{3},\mathbf{n})$. We will look first for triangle curves – or equivalently, triples of generators – of type $(2,3,n)$. Let ϕ denote Euler's phi function.

Lemma 1.4. *Let p be a prime number $p \geq 5$ and n any natural number dividing either $(p-1)/2$ or $(p+1)/2$.*

(i) *There are $\phi(n)/2$ conjugacy classes of elements of order n in $\mathrm{PSL}(2,p)$.*
(ii) *These are characterized by the trace of any of its elements.*
(iii) *In fact for every $\mathfrak{c} \in \mathrm{PSL}(2,p)$ of order n, the elements $\mathfrak{c}^i$ with $\gcd(i,n) = 1$ and $0 < i < n$, provide representatives for all these conjugacy classes; the elements $\mathfrak{c}^i$ and $\mathfrak{c}^{n-i}$ lying in the same class.*

Proof. The group $\mathrm{PSL}(2,p)$ contains two cyclic subgroups of order $(p-1)/2$ and $(p+1)/2$, namely the projective image of the subgroups of $\mathrm{SL}(2,p)$

$$H_- = \left\{ M_\lambda \equiv \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \;:\; \lambda \in \mathbb{F}_p^* \right\} \cong \mathbb{F}_p^*$$

and

$$H_+ = \left\{ M_{(x,y)} \equiv \begin{pmatrix} x & \varepsilon y \\ y & x \end{pmatrix} \,:\, x, y \in \mathbb{F}_p^*,\ x^2 - \varepsilon y^2 = 1 \right\},$$

where ε is a generator of the cyclic group $\mathbb{F}_p^*$ (see for instance [27], §5.2).

Now, every element of $\mathrm{PSL}(2,p)$ of order n dividing $(p-1)/2$ (resp. $(p+1)/2$) is conjugate to an element of H_- (resp. H_+), which contains $\phi(n)$ such elements of order n. All these matrices have different traces $\lambda_i + \lambda_i^{-1}$ (resp. $2x$) except for mutually inverse elements M_{λ_i} and $M_{\lambda_i^{-1}}$ (resp. $M_{(x,y)}$ and $M_{(x,-y)}$), which are therefore conjugate. It follows that there are $\phi(n)/2$ conjugacy classes of elements of order n in $\mathrm{PSL}(2,p)$.

Point (iii) follows from the fact that H_- (resp. H_+) is cyclic. □

We are now interested in the number of classes of triples of generators of $G = \mathrm{PSL}(2,p)$ of type $(2,3,n)$ under the action $\mathrm{Aut}(G)$. Recall that elements of order 2 and 3 in $\mathrm{PSL}(2,p)$ have trace 0 and ± 1 respectively.

Lemma 1.5. *Let p be a prime number $p \geq 5$ and $n > 6$ any natural number dividing either $(p-1)/2$ or $(p+1)/2$.*

(i) *There are $\phi(n)$ classes of triples of generators of type $(2,3,n)$ modulo $\mathrm{I}(G;2,3,n) = G$.*
(ii) *There are $\phi(n)/2$ classes of triples of generators of type $(2,3,n)$ modulo $\mathrm{A}(G;2,3,n) = \mathrm{Aut}(G) \cong \mathrm{PGL}(2,p)$.*
(iii) *The $\phi(n)/2$ classes modulo $\mathrm{Aut}(G)$ can be represented by triples of the form $(\mathfrak{a}_i, \mathfrak{b}_i, \mathfrak{c}^i)$, where $\mathfrak{c}$ is an element of order n, $\mathfrak{a}_i$ and $\mathfrak{b}_i$ are suitable elements of order 2 and 3, respectively, and $1 \leq i < n/2$ with $\gcd(i,n) = 1$. These, together with another set of $\phi(n)/2$ triples $(\mathfrak{a}_i', \mathfrak{b}_i', \mathfrak{c}^i)$ of the same form, provide representatives for the $\phi(n)$ classes modulo G.*
(iv) *The conjugacy class of the element $\mathfrak{c}^i$ of order n characterizes the conjugacy class of the triple modulo $\mathrm{Aut}(G)$.*

Proof. We know that there are $\phi(n)/2$ conjugacy classes of elements of order n. For each class $\mathcal{C}$ let $t \in \mathbb{F}_p$ be the trace of any element $\mathfrak{c} \in \mathcal{C}$, which is defined up to multiplication by ± 1. The possible traces of triples of type $(2,3,n)$ are therefore $(0, \pm 1, \pm t)$. For all of them the discriminant $t^2 - 3$ is different from zero, since otherwise the order of $\mathfrak{c}$ would be less than or equal to 6. Indeed, by the Cayley–Hamilton theorem $\mathfrak{c}^2 - t\mathfrak{c} + \mathrm{Id} = 0$, and therefore we would have

$$0 = (\mathfrak{c}^2 - t\mathfrak{c} + \mathrm{Id})^2 - (2 + 2\mathfrak{c}^2)(\mathfrak{c}^2 - t\mathfrak{c} + \mathrm{Id}) = -\mathfrak{c}^4 + \mathfrak{c}^2 - \mathrm{Id}\,,$$

which implies

$$0 = \mathfrak{c}^2(-\mathfrak{c}^4 + \mathfrak{c}^2 - \mathrm{Id}) + (-\mathfrak{c}^4 + \mathfrak{c}^2 - \mathrm{Id}) = -\mathfrak{c}^6 + \mathfrak{c}^4 - \mathfrak{c}^2 - \mathfrak{c}^4 + \mathfrak{c}^2 - \mathrm{Id} = -\mathfrak{c}^6 - \mathrm{Id}\,,$$

hence $\mathfrak{c}^6 = \mathrm{Id}$ in $\mathrm{PSL}(2,p)$.

Now by Lemma 1.3 it is enough to study $E(0,1,t)$ and, since $(0,1,t)$ is neither singular nor exceptional, the result follows from Theorem 1.1. □

By the previous two lemmas, for any element $\mathfrak{c}$ of order n the $\phi(n)/2$ conjugacy classes of triples of type $(2,3,n)$ have representatives $(\mathfrak{a}_i, \mathfrak{b}_i, \mathfrak{c}^i)$, where $1 \leq i < n/2$ with $\gcd(i,n) = 1$. Let us denote by (E_i, f_i) the corresponding $G-$covers. The curves E_i are pairwise non-isomorphic. This can be seen as follows: suppose that we had $E_i \cong E_j$ and set $\Delta = \Delta(2,3,n)$. Then their uniformising groups $K_i \lhd \Delta$ and $K_j \lhd \Delta$ would be conjugate by an element of $\mathrm{PSL}(2,\mathbb{R})$, say $K_j = \alpha K_i \alpha^{-1}$.

Note that α does not belong to $\Delta(2,3,n)$ since the triples defining the $G-$coverings (E_i, f_i) and (E_j, j_j) are not equivalent modulo G. Conjugating now the inclusion $K_j \lhd \Delta$ by α^{-1} we get $\alpha^{-1}K_j\alpha = K_i \lhd \alpha^{-1}\Delta\alpha$. But then K_i is normal in both Δ and $\alpha^{-1}\Delta\alpha$. Since $\Delta(2,3,n)$ is a maximal Fuchsian group this is impossible unless $\alpha \in \Delta(2,3,n)$, which is a contradiction.

We claim now that for any k with $\gcd(n,k) = 1$, the curves E_1 and E_k are Galois conjugate. The idea of the proof is contained in the case $n = 7$, proved by M. Streit in [**63**].

Let us consider the action on (E_1, f_1) of an element $\sigma_k \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ such that $\sigma_k(\zeta_n) = \zeta_n^{k^{-1}}$. By Proposition 1.3 the $G-$covering (E_1^σ, f_1^σ) must correspond to a triple $(h_{\mathfrak{a}}\mathfrak{a}^{\alpha'}h_{\mathfrak{a}}^{-1}, h_{\mathfrak{b}}\mathfrak{b}^{\beta'}h_{\mathfrak{b}}^{-1}, \mathfrak{c}^{\gamma'})$, with $\gamma' \equiv k \pmod n$. By the previous lemma this triple is equivalent to $(\mathfrak{a}_k, \mathfrak{b}_k, \mathfrak{c}^k)$, and so $(E_1^\sigma, f_1^\sigma) = (E_k, f_k)$. There are therefore $\phi(n)$ options for k, yielding $\phi(n)/2$ different curves Galois conjugate to E_1. This is because for each such k the curves $E_1^{\sigma_k}$ and $E_1^{\sigma_{n-k}}$ are isomorphic since, $\mathfrak{c}^k$ and $\mathfrak{c}^{n-k}$ being conjugate, they correspond to equivalent triples.

Hence the $G-$coverings (E_i, f_i) form a complete orbit under the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Finally note that if $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ is the complex conjugation $\sigma(\mathrm{w}) = \overline{\mathrm{w}}$ then $\sigma(\zeta_n) = \zeta_n^{-1}$, and since $\mathfrak{c}_1$ and $\mathfrak{c}_1^{-1}$ are conjugate in $\mathrm{PSL}(2,p)$ then $E_1^\sigma \cong E_1$. From this fact one can conclude that $\mathbb{Q}(\zeta_n) \cap \mathbb{R} = \mathbb{Q}(\cos \pi/n)$ is the field of moduli of these curves, and hence a field of definition ([**63**]).

We have proved the following theorem.

Theorem 1.2. *Let p be a prime number $p \geq 5$ and $n > 6$ any natural number dividing either $(p-1)/2$ or $(p+1)/2$.*

(i) *The $\phi(n)/2$ covers (E_i, f_i), for $1 \leq i < n/2$ and $\gcd(i,n) = 1$, are the only $G-$coverings with covering group $G = \mathrm{PSL}(2,p)$ and type $(2,3,n)$.*
(ii) *They correspond to the triples $(\mathfrak{a}_i, \mathfrak{b}_i, \mathfrak{c}^i)$.*
(iii) *They form a complete orbit under the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.*
(iv) *The curves E_i have genus $g = \frac{1}{24n}(n-6)p(p-1)(p+1)+1$ and they are pairwise non-isomorphic. They can all be defined over $\mathbb{Q}(\cos(\pi/n))$ and have automorphism group $\mathrm{Aut}(E_i) \cong G$.*

The expression for the genus is a consequence of the Riemann–Hurwitz formula and the claim about the automorphism group follows from the fact that $\Delta(2,3,n)$ is a maximal Fuchsian group ([**61**]).

Example 1.1. For $p = 13$ and $n = 7$ the following triples define three Galois conjugate curves of type $(2,3,7)$:

$$
\begin{aligned}
(a_1, b_1, c) &= \left(\begin{pmatrix} 8 & 3 \\ 0 & 5 \end{pmatrix}, \begin{pmatrix} 1 & 8 \\ 8 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 12 & 6 \end{pmatrix} \right), \\
(a_2, b_2, c^2) &= \left(\begin{pmatrix} 0 & 12 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 6 & 12 \\ 4 & 6 \end{pmatrix}, \begin{pmatrix} 12 & 6 \\ 7 & 9 \end{pmatrix} \right), \\
(a_3, b_3, c^3) &= \left(\begin{pmatrix} 12 & 1 \\ 11 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 10 \\ 9 & 1 \end{pmatrix}, \begin{pmatrix} 7 & 9 \\ 4 & 9 \end{pmatrix} \right).
\end{aligned}
$$

Any other triple $(\mathfrak{a}', \mathfrak{b}', \mathfrak{c}')$ of type $(2,3,7)$ can be mapped by an automorphism of $\mathrm{PSL}(2,13)$ to one of these, depending on the conjugacy class of $\mathfrak{c}'$. These three curves are Hurwitz curves of genus 14, i.e. curves S whose automorphism group reaches the Hurwitz bound $|\mathrm{Aut}(S)| \le 84(g-1)$. They are defined over the number field $\mathbb{Q}(\cos \pi/7)$ and they are Galois conjugate under the action of any Galois element satisfying $\zeta_7 \mapsto \zeta_7^2$ and $\zeta_7 \mapsto \zeta_7^3$ respectively ([63]).

1.5.2. Type $(\mathbf{p}, \mathbf{p}, \mathbf{p})$. We focus now on triples of type (p,p,p) in the groups $G = \mathrm{PSL}(2,p)$ for $p > 5$.

Lemma 1.6. *Let $p > 5$ be a prime number.*

(i) *There is only one class of triples of generators of type (p,p,p) modulo $\mathrm{Aut}(G) < \mathrm{A}(G;p,p,p)$, which is represented by*

$$u = \begin{pmatrix} -1 & 1 \\ -4 & 3 \end{pmatrix}, \quad v = \begin{pmatrix} 3 & -4 \\ 4 & -5 \end{pmatrix}, \quad w = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

(ii) *Modulo $\mathrm{I}(G;p,p,p)$ there are two classes of triples of generators of type (p,p,p), represented by elements of the form (u,v,w) and (u',v',w^{ε}), where $(\frac{\varepsilon}{p}) = -1$, i.e. ε is not a quadratic residue modulo p and there exists $\psi \in \mathrm{Aut}(G) \setminus \mathrm{Inn}(G)$ such that $(u',v',w^{\varepsilon}) = \psi(u,v,w)$.*

Proof. It can be easily checked that u, v, w are elements of order p whose product is the identity. Moreover, recall that all triples of type (p,p,p) have traces of the form $(\pm 2, \pm 2, \pm 2)$. By Lemma 1.3 we can consider just the cases $(2,2,2)$ and $(2,-2,2)$, but only the latter is neither singular nor exceptional, and therefore it follows from Theorem 1.1 that (u,v,w) is the only triple of generators of type (p,p,p) modulo $\mathrm{Aut}(G)$.

It also follows from the same theorem that there are two such triples of generators modulo G and, since for any ε which is not a quadratic residue modulo p the element w^{ε} is not conjugate to w, we can suppose that these two classes of triples of generators are represented by (u,v,w) and (u',v',w^{ε}). □

Now take the $G-$covering (E,f) corresponding to the triple of generators (u,v,w) above. Lemma 1.6 implies that $(E,f) \cong (E^{\sigma}, f^{\sigma})$ for any $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. This means that the field of moduli of E is $\mathbb{Q}$, and since E is a triangle curve, $\mathbb{Q}$ is a field of definition as well.

Theorem 1.3. *For each prime number $p > 5$ there is a unique $G-$covering (E,f) of type (p,p,p) with $G = \mathrm{PSL}(2,p)$. Moreover the following properties hold:*

(i) *the $G-$covering (E,f) can be defined over $\mathbb{Q}$;*
(ii) *E has genus $g = \frac{1}{4}(p+1)(p-1)(p-3)+1$;*
(iii) *the automorphism group $\mathrm{Aut}(E)$ is isomorphic to $\mathrm{PSL}(2,p) \times \mathfrak{S}_3$.*

Proof. The formula for the genus is a consequence of the Riemann–Hurwitz formula. After the comment preceding the statement of the theorem the only part left to prove is the one regarding the automorphism group. Let K be the Fuchsian group uniformising the curve E, i.e. the kernel of the epimorphism $\rho : \Delta(p,p,p) \longrightarrow \mathrm{PSL}(2,p)$ defined by

$$\begin{array}{rccc} \rho : & \Delta(p,p,p) & \longrightarrow & \mathrm{PSL}(2,p) \\ & x & \longmapsto & u \\ & y & \longmapsto & v \\ & z & \longmapsto & w \end{array}$$

where x, y, z are the generators of $\Delta(p,p,p)$ chosen in (1.1), and u, v, w are as in Lemma 1.6.

It is well-known that the group $\Delta(p,p,p)$ injects into the maximal triangle group $\Delta(2,3,2p)$ as a normal subgroup of index 6. This injection can be realized geometrically as the inclusion map of $\Delta(p,p,p)$ in the triangle group $\widetilde{\Delta}(2,3,2p)$ associated to one of the six triangles $\widetilde{T} = \widetilde{T}(2,3,2p)$ of angles $\pi/2$, $\pi/3$, $\pi/2p$ in which $T(p,p,p)$ is naturally subdivided (see Figure 1.3). Note that $\widetilde{T} = \alpha(T)$, and hence $\widetilde{\Delta}(2,3,2p) = \alpha\Delta(2,3,2p)\alpha^{-1}$, for some $\alpha \in \mathrm{PSL}(2,\mathbb{R})$.

Now we consider the group homomorphism defined by

$$\begin{array}{rccl} \widetilde{\rho}: & \widetilde{\Delta}(2,3,2p) & \longrightarrow & \mathrm{PSL}(2,p)\times\mathfrak{S}_3 \\ & \tilde{x} & \longmapsto & x' = (X,\mu) \\ & \tilde{y} & \longmapsto & y' = (Y,\nu) \\ & \tilde{z} & \longmapsto & z' = (Z,\mu\nu) \end{array}$$

where:

- $\tilde{x}$, $\tilde{y}$, $\tilde{z}$ are the generators of $\widetilde{\Delta}(2,3,2p)$ of orders 2, 3 and $2p$ depicted in Figure 1.3;
- $X = \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix}$, $Y = \begin{pmatrix} -1 & \frac{p+3}{2} \\ -2 & 2 \end{pmatrix}$ and $Z = \begin{pmatrix} 1 & \frac{p+1}{2} \\ 0 & 1 \end{pmatrix}$;
- μ, ν are generators of $\mathfrak{S}_3$ such that $\mu^2 = \nu^3 = (\mu\nu)^2 = 1$.

Notice that the generators x, y, z of $\Delta(p,p,p)$ are related to the generators $\tilde{x}$, $\tilde{y}$, $\tilde{z}$ of $\widetilde{\Delta}(2,3,2p)$ by

$$\begin{array}{rcl} x & = & \tilde{y}\tilde{z}^2\tilde{y}^{-1} = \tilde{x}\tilde{z}^2\tilde{x}^{-1}\,, \\ y & = & \tilde{y}^{-1}\tilde{z}^2\tilde{y}\,, \\ z & = & \tilde{z}^2\,. \end{array}$$

This can be seen by checking that the fixed points of $\tilde{z}^2$, $\tilde{y}\tilde{z}\tilde{y}^{-1}$ and $\tilde{y}^{-1}\tilde{z}^2\tilde{y}$ are v_∞, $\tilde{y}(v_\infty) = v_0$ and $\tilde{y}^{-1}(v_\infty) = v_1$ respectively (see Figure 1.3).

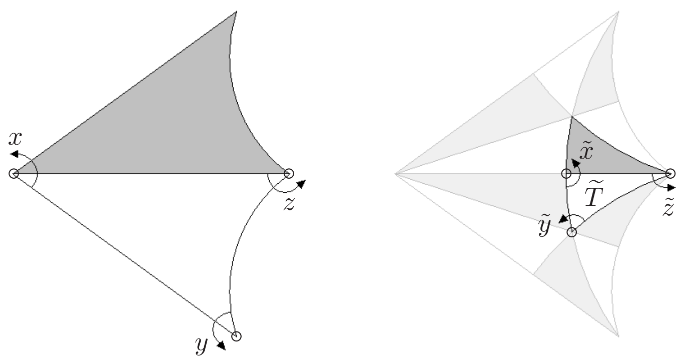


Figure 1.3. Fundamental domains and generators for the groups $\Gamma(p,p,p)$ and $\widetilde{\Gamma}(2,3,2p)$ $(p=5)$.

Now we point out the following facts:

- The rule $\widetilde{\rho}$ certainly defines a homomorphism, since $\mathrm{ord}(x') = 2$, $\mathrm{ord}(y') = 3$, $\mathrm{ord}(z') = 2p$ and $x'y'z' = \mathrm{Id}$.

- The restriction of $\widetilde{\rho}$ to $\Delta = \Delta(p,p,p)$ coincides with ρ. This is because of the following identities:

$$\begin{aligned} \widetilde{\rho}(x) &= y'z'^2y'^{-1} = x'z'^2x'^{-1} = (u, \mathrm{Id}) \\ \widetilde{\rho}(y) &= y'^{-1}z'^2y' = (v, \mathrm{Id}) \\ \widetilde{\rho}(z) &= z'^2 = (w, \mathrm{Id}) \end{aligned}$$

As a consequence $\widetilde{\rho}$ is an epimorphism. In fact it is easy to see that the subgroup $\widetilde{\rho}(\Delta(p,p,p)) = G$ together with the elements $\widetilde{\rho}(\tilde{x}) = x'$ and $\widetilde{\rho}(\tilde{y}) = y'$ already generate a group in which G has index at least 6.
- In particular $K < \ker(\widetilde{\rho})$ and since

$$[\widetilde{\Delta}(2,3,2p) : \Delta(p,p,p)] = [\mathrm{PSL}(2,p) \times \mathfrak{S}_3 : \mathrm{PSL}(2,p)]\,,$$

it follows that $K = \ker(\widetilde{\rho})$. Moreover, since $\widetilde{\Delta}(2,3,2p)$ is a maximal triangle group it also follows that $\widetilde{\Delta}(2,3,2p)$ equals $N(K)$, the normaliser of K in $\mathrm{PSL}(2,\mathbb{R})$.

We conclude that $\mathrm{Aut}(E) \cong N(K)/K \cong \mathrm{PSL}(2,p) \times \mathfrak{S}_3$. □

The general study of the extendability of the automorphism group of triangle curves has been considered by Bujalance, Cirre and Conder (see [13], Thm. 5.2).

Example 1.2. In the particular case $p = 7$ the two conjugacy classes of triples of type $(7,7,7)$ are represented by

$$u = \begin{pmatrix} 6 & 1 \\ 3 & 3 \end{pmatrix}, \quad v = \begin{pmatrix} 3 & 3 \\ 4 & 2 \end{pmatrix}, \quad w = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

and

$$u^{-1} = \begin{pmatrix} 3 & 6 \\ 4 & 6 \end{pmatrix}, \quad v' = \begin{pmatrix} 6 & 0 \\ 3 & 6 \end{pmatrix}, \quad w^{-1} = \begin{pmatrix} 1 & 6 \\ 0 & 1 \end{pmatrix},$$

which are conjugate under the element $\alpha = \left(\begin{smallmatrix} 6 & 1 \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{PGL}(2,7) \cong \mathrm{Aut}(\mathrm{PSL}(2,7))$.

We will write (D, f) for the corresponding $G-$covering.

1.5.3. Type $(3,3,4)$ in $\mathrm{PSL}(2,7)$. We will focus our attention now on triples of type $(3,3,4)$ in $G = \mathrm{PSL}(2,7)$. It can be found by computational means (e.g. with MAGMA) that up to conjugation in $\mathrm{PSL}(2,7)$ there are four such triples, namely

$$\begin{aligned} (a_1, b_1, c) &= \left(\begin{pmatrix} 1 & 5 \\ 4 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 3 & 4 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 6 & 3 \end{pmatrix} \right), \\ (a_2, b_2, c) &= \left(\begin{pmatrix} 0 & 6 \\ 1 & 6 \end{pmatrix}, \begin{pmatrix} 5 & 1 \\ 4 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 6 & 3 \end{pmatrix} \right), \\ (a_1', b_1', c) &= \left(\begin{pmatrix} 1 & 3 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 4 & 0 \\ 2 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 6 & 3 \end{pmatrix} \right), \\ (a_2', b_2', c) &= \left(\begin{pmatrix} 5 & 1 \\ 4 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 6 \\ 0 & 4 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 6 & 3 \end{pmatrix} \right). \end{aligned}$$

On the other hand, in $\mathrm{PGL}(2,7) \cong \mathrm{Aut}(G)$ there are two non-equivalent triples of type $(2,3,8)$, namely

$$\begin{aligned}(r_1,s_1,t_1) &= \left(\begin{pmatrix}3&3\\2&4\end{pmatrix},\begin{pmatrix}2&0\\3&4\end{pmatrix},\begin{pmatrix}5&5\\2&6\end{pmatrix}\right),\\ (r_2,s_2,t_2=t_1^5) &= \left(\begin{pmatrix}2&0\\4&5\end{pmatrix},\begin{pmatrix}6&4\\2&4\end{pmatrix},\begin{pmatrix}6&1\\6&2\end{pmatrix}\right).\end{aligned}$$

Parts (iii), (iv) and (v) of the following theorem are contained in a paper by M. Conder, G. Jones, M. Streit and J. Wolfart ([18]) and the two remaining ones could be easily deduced from them. Since they consider a wide range of groups and types, their methods are much more sophisticated than ours, so we provide here an *ad hoc* proof for the case we are interested in.

THEOREM 1.4. *The following statements hold.*

(i) *The $G-$coverings (D_1,f_1) and (D_2,f_2), defined by the triples (a_1,b_1,c) and (a_2,b_2,c) respectively, are the only two $G-$coverings of type $(3,3,4)$ and covering group $\mathrm{PSL}(2,7)$, up to isomorphism.*

(ii) *The $G-$coverings (D_1',h_1) and (D_2',h_2), defined by the triples (r_1,s_1,t_1) and (r_2,s_2,t_2) respectively, are the only two $G-$coverings of type $(2,3,8)$ and covering group $\mathrm{PGL}(2,7)$, up to isomorphism. Moreover, D_1' and D_2' are non-isomorphic curves.*

(iii) *$D_1 \cong D_1'$ and $D_2 \cong D_2'$. In particular D_1 and D_2 are not isomorphic. Both curves have genus 49.*

(iv) *Let $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ satisfy $\sigma(\zeta_8)=\zeta_8^5$. Then $D_1^\sigma = D_2$.*

(v) *D_1 and D_2 are defined over $\mathbb{Q}(\sqrt{2})$. In particular $\overline{D}_1 \cong D_1$ and $\overline{D}_2 \cong D_2$.*

PROOF. *(i)* The triples (a_1,b_1,c) and (a_1',b_1',c) (resp. (a_2,b_2,c) and (a_2',b_2',c)) are conjugate by the element $\left(\begin{smallmatrix}4&5\\2&5\end{smallmatrix}\right)$ (resp. $\left(\begin{smallmatrix}1&6\\1&5\end{smallmatrix}\right)$), so they are equivalent under the action of $\mathrm{PGL}(2,7) \cong \mathrm{Aut}(\mathrm{PSL}(2,7))$. However (a_1,b_1,c) and (a_2,b_2,c) are not conjugate in $\mathrm{PGL}(2,7)$.

(ii) The $G-$coverings D_1' and D_2' correspond to the inclusion of certain surface normal subgroups $K_1, K_2 < \Delta(2,3,8)$. We claim that not even the curves D_1' and D_2' are isomorphic. If they were there would exist an $\alpha \in \mathrm{PSL}(2,\mathbb{R})$ such that $K_2 = \alpha K_1 \alpha^{-1}$. But then K_2 would be normal both in $\Delta(2,3,8)$ and $\alpha\Delta(2,3,8)\alpha^{-1}$, and since $\Delta(2,3,8)$ is a maximal Fuchsian group ([61]) this can only occur if $\alpha \in \Delta(2,3,8)$. But K_1 and K_2 are not conjugate in $\Delta(2,3,8)$ because their corresponding defining triples are not equivalent.

(iii) In a way similar to the case of $\Delta(p,p,p) < \widetilde{\Delta}(2,3,2p)$ in the proof of Theorem 1.3, the group $\Delta(3,3,4)$ is included in the triangle group $\widetilde{\Delta}(2,3,8)$ associated to the triangle $\widetilde{T} = \widetilde{T}(2,3,8)$ in Figure 1.4. Again $\widetilde{T} = \alpha(T)$ for some $\alpha \in \mathrm{PSL}(2,\mathbb{R})$, and hence $\widetilde{\Delta}(2,3,8) = \alpha\Delta(2,3,8)\alpha^{-1}$.

Now consider the following diagram

$$\begin{array}{ccccc}\widetilde{K} & \hookrightarrow & \widetilde{\Delta}(2,3,8) & \xrightarrow{\widetilde{\rho}} & \mathrm{PGL}(2,7)\\ & & \uparrow & & \uparrow\\ K & \hookrightarrow & \Delta(3,3,4) & \xrightarrow{\rho} & \mathrm{PSL}(2,7)\end{array}$$

where the vertical arrows are the natural inclusions, $\widetilde{K} = \ker \widetilde{\rho}$, $K = \ker \rho$ and the two epimorphisms ρ and $\widetilde{\rho}$ are given by

$$\begin{array}{rcccc} \rho: & \Delta(3,3,4) & \longrightarrow & \mathrm{PSL}(2,7) \\ & x & \longmapsto & a_1 \\ & y & \longmapsto & b_1 \\ & z & \longmapsto & c \end{array} \qquad \begin{array}{rcccc} \widetilde{\rho}: & \widetilde{\Delta}(2,3,8) & \longrightarrow & \mathrm{PGL}(2,7) \\ & \tilde{x} & \longmapsto & r_1 \\ & \tilde{y} & \longmapsto & s_1 \\ & \tilde{z} & \longmapsto & t_1 \end{array}$$

where x, y, z and $\tilde{x}, \tilde{y}, \tilde{z}$ are the generators of $\Delta(3,3,4)$ and $\widetilde{\Delta}(2,3,8)$ respectively provided by the rotations depicted in Figure 1.4 below.

The following obvious identities show that this is a commutative diagram:

$$\begin{array}{lclcl} y = \tilde{y} & , & z = \tilde{z}^2 & & \text{(see Figure 1.4)} \quad \text{and} \\ b_1 = s_1 & , & c = t_1^2 & & \text{in } \mathrm{PGL}(2,7). \end{array}$$

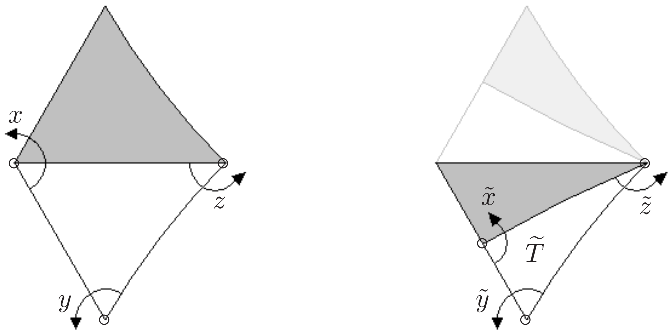


FIGURE 1.4. Fundamental domains and generators for the groups $\Delta(3,3,4)$ and $\widetilde{\Delta}(2,3,8)$.

Therefore it is clear that $K = \widetilde{K} \cap \Delta(3,3,4)$. Now since $[\widetilde{\Delta}(2,3,8) : \Delta(3,3,4)]$ equals $[\mathrm{PGL}(2,7) : \mathrm{PSL}(2,7)]$ it follows that $\widetilde{K} = K$ and $D_1 \cong D_1'$. It can be argued in the same way to deduce that $D_2 \cong D_2'$. Since we have already proved that $D_1' \not\cong D_2'$, this implies $D_1 \not\cong D_2$.

The statement about the genus follows from the Riemann–Hurwitz formula.

(iv) We note now that the conjugacy classes of (r_1, s_1, t_1) and (r_2, s_2, t_2) in $\mathrm{PGL}(2,7)$ are determined by the conjugacy classes in $\mathrm{PGL}(2,7)$ of their elements of order 8 (t_1 and $t_2 = t_1^5$ respectively). Therefore applying Proposition 1.3 with an element $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ such that $\sigma(\zeta_8) = \zeta_8^5$ we can conclude that $D_1'^{\sigma} \cong D_2'$, and therefore $D_1^{\sigma} \cong D_2$.

(v) We have already mentioned the fact that triangle curves are defined over their field of moduli. By the comments in the proof of the previous point, any Galois element fixing the field $\mathbb{Q}(\zeta_8)$ belongs to the inertia groups I_{D_1} and I_{D_2}. Moreover, by Proposition 1.4 the curve $\overline{D'}_1 = \overline{D}_1$ (resp. $\overline{D'}_2 = \overline{D}_2$) is defined by the triple $(r_1^{-1}, r_1 s_1^{-1} r_1^{-1}, t_1^{-1})$ (resp. $(r_2^{-1}, r_2 s_2^{-1} r_2^{-1}, t_2^{-1})$). Since t_1 and t_1^{-1} (resp. t_2 and t_2^{-1}) lie in the same conjugacy class, we deduce that $\overline{D}_1 \cong D_1$ (resp. $\overline{D}_2 \cong D_2$), and so complex conjugation belongs also to both inertia groups.

As a consequence the field of moduli of both D_1 and D_2 is contained in $\mathbb{Q}(\zeta_8) \cap \mathbb{R} = \mathbb{Q}(\sqrt{2})$. Since by points (iii) and (iv) this field must be a non-trivial extension of $\mathbb{Q}$, we deduce that $\mathbb{Q}(\sqrt{2})$ is the field of moduli, hence the minimum field of definition of both D_1 and D_2. □

REMARK 1.3. Point (v) explains why, although the curves D_1 and D_2 are determined by $(3, 3, 4)$ triples, in order to distinguish them one needs to work with triples of type $(2, 3, 8)$. Since the action of a Galois element σ on ζ_3 and ζ_4 does not determine $\sigma(\sqrt{2})$, the effect of Galois conjugation could not be seen in the $(3, 3, 4)$ triples.

CHAPTER 2

Uniform dessins

"Arithmétique! Algèbre! Géométrie! Trinité grandiose! Triangle lumineux! Celui qui ne vous a pas connues est un insensé!"

— COMTE DE LAUTRÉAMONT, Les chants de Maldoror

The correspondence between (equivalence classes of) dessins d'enfants and (isomorphism classes of) algebraic curves defined over $\overline{\mathbb{Q}}$ is not bijective, given a Riemann surface defined over $\overline{\mathbb{Q}}$, there are many different dessins d'enfants on S. However, the question of when two different dessins d'enfants live on the same surface is too wide to answer in its full generality, so one has to restrict to certain families of dessins.

In [35] it was considered the case of regular dessins of the same type (see also [30]). Let us remind that a regular dessin of type (l,m,n) on a surface S arises as the normal inclusion of a group K uniformising S in a triangle group $\Delta(l,m,n)$ and, therefore, the situation of several regular dessins of the same type on S corresponds to the normal inclusion of K in different conjugate triangle groups of type (l,m,n). Girondo and Wolfart proved that if this happens, these inclusions are induced by inclusions between triangle groups.

The next family of dessins that one could study is that of uniform dessins. Recall that a uniform dessin of type (l,m,n) on a surface S arises as the inclusion – not necessarily normal – of a group K uniformising S in a triangle group $\Delta(l,m,n)$. As a consequence, the existence of several uniform dessins of type (l,m,n) corresponds to the inclusion of K in different triangle groups of type (l,m,n)

To put the problem in a precise form we observe first that a surface group K contained in a triangle group Δ is contained in all triangle groups Δ' containing Δ (and maybe also in some triangle subgroups of Δ), all these inclusions inducing dessins of different types on the surface S. All possibilities of such inclusions are well known by work of Singerman [61], so one can concentrate on dessins of the same type (l,m,n), i.e. on the following question. Let K be a Fuchsian surface group contained in a triangle group $\Delta(l,m,n)$: which and how many different conjugate groups $\alpha\Delta\alpha^{-1}$, $\alpha \in \mathrm{PSL}(2,\mathbb{R})$, contain K as well?

Note that one could consider the following similar question. Let Δ be a Fuchsian triangle group and let K be a finite index subgroup: for which and for how many $\alpha \in \mathrm{PSL}(2,\mathbb{R})$ do we have $\alpha^{-1}K\alpha < \Delta$?

These two questions are not equivalent only when α belongs either to $N(\Delta)$ or to $N(K)$, the normalisers of Δ and K in $\mathrm{PSL}(2,\mathbb{R})$. However, if $\alpha \in N(\Delta)$ the two inclusions $K, \alpha^{-1}K\alpha < \Delta$ correspond to renormalised dessins, and if $\alpha \in N(K)$, conjugation by α induces an isomorphism of the curve, which in turn induces an isomorphism between the dessins corresponding to $K < \Delta$ and $K < \alpha\Delta\alpha^{-1}$. Therefore, if one wants to study the problem of non-isomorphic dessins of the same type on the same surface not related by renormalisation, then both questions are interchangeable. We will thus focus on the first version, which is more natural. Moreover, since conjugation of K by an element of $N(\Delta)$ induces a renormalisation of the dessin, we will only count residue classes $\alpha \in \mathrm{PSL}(2,\mathbb{R})/N(\Delta)$. Let us stress here that, in view of the question we are dealing with, two dessins are considered different if they correspond to different Belyi functions. Whether they are isomorphic or not is a completely different question.

If K is included in both Δ and $\alpha\Delta\alpha^{-1}$, the element α belongs by definition to both the commensurator groups of K and of Δ. Now, by the theorem of Margulis (see section 0.7) the commensurator $\overline{\Delta} = \mathrm{Comm}(\Delta)$ of a non-arithmetic Fuchsian group Δ is a finite extension of Δ and a Fuchsian group itself. But finite extensions of triangle groups are known to be triangle groups again, so if Δ is non-arithmetic, $\overline{\Delta}$ is itself a triangle group, and consulting Takeuchi's list of arithmetic triangle groups [64] and Singerman's list of inclusion relations [61] it is easy to see that the index $[\overline{\Delta} : \Delta]$ is at most 6. So we have the first part of the following theorem.

Theorem 2.1. *Surface groups contained in a non-arithmetic Fuchsian triangle group Δ define isomorphic surfaces if and only if they are conjugate in a maximal Fuchsian triangle group $\overline{\Delta}$ extending Δ. They fall in at most 6 different conjugacy classes under conjugation by Δ. If K is such a surface group then the number of triangle groups conjugate to Δ in which K is included is 1, 3 or 4.*

Proof. The second part of the theorem follows from the fact that non-normal inclusions $\Delta < \overline{\Delta}$ of non-arithmetic triangle groups occur only with index 3 for $\Delta(2,n,2n) < \Delta(2,3,2n)$, or 4 for $\Delta(3,n,3n) < \Delta(2,3,3n)$. □

2.1. Arithmetic surface groups, localisation

Now we concentrate on the remaining case that K and Δ are arithmetic Fuchsian groups, i.e. they are commensurable to a norm 1 group $\mathcal{M}^1$ of a maximal order $\mathcal{M}$ in a quaternion algebra A defined over a totally real number field k and having precisely one embedding into the matrix algebra $M_2(\mathbb{R})$. The situation here is quite different, as indicated already by the analogous question for normal subgroups in Theorem 3 of [35]. By [64] we know which triangle groups can be identified with the norm 1 group of a maximal order, and most of the arguments will be applied to these cases.

Since we have to work in the quaternion algebra A it is often necessary to replace all Fuchsian groups K above with their preimages $\hat{K}$ in $\mathrm{SL}(2,\mathbb{R})$. However, if it is clear from the context where the groups are situated, we will often omit the hat to simplify the notation.

We consider now the norm 1 group $\mathcal{M}^1$ (which in most cases is a triangle group itself [64]) and restrict our attention to common finite index subgroups K of $\mathcal{M}^1$ and $\beta^{-1}\mathcal{M}^1\beta$ and the possible conjugators β in this configuration. Clearly, conjugation by such a β induces an automorphism of the quaternion algebra, therefore the Skolem–Noether theorem allows to replace β with a more convenient element $\alpha \in A$. By multiplication with a denominator in the integers of k we can even suppose α to be in the maximal order $\mathcal{M}$.

THEOREM 2.2. *Let $\mathcal{M}^1$ be the norm 1 group of a maximal order $\mathcal{M}$ and suppose that $\beta \in \mathrm{SL}(2,\mathbb{R})$ is such that $\mathcal{M}^1 \cap \beta^{-1}\mathcal{M}^1\beta$ has finite index in $\mathcal{M}^1$ and $\beta^{-1}\mathcal{M}^1\beta$. Then β can be replaced with a scalar multiple $\alpha \in \mathrm{GL}(2,\mathbb{R})^+ \cap \mathcal{M} \subset A$.*

Under these conditions $\mathcal{M}^1 \cap \alpha^{-1}\mathcal{M}^1\alpha$ is the norm 1 group of an Eichler order $\mathcal{M} \cap \alpha^{-1}\mathcal{M}\alpha$. The index of $\mathcal{M}^1 \cap \alpha^{-1}\mathcal{M}^1\alpha$ in $\mathcal{M}^1$ gives a lower bound for $[\mathcal{M}^1 : K]$ where K denotes a surface group contained in both $\mathcal{M}^1$ and $\alpha^{-1}\mathcal{M}^1\alpha$.

For arithmetic triangle groups one has the additional advantage that all quaternion algebras in question have type number 1 ([64], Prop. 3), therefore all maximal orders are conjugate in A and all Eichler orders are intersections of conjugate maximal orders. So counting multiple dessins on $\mathbb{H}/K$ amounts to counting maximal orders containing $\hat{K}$.

Maximal orders are easier to classify locally, i.e. over local fields, and the type number 1 property implies that there are bijections between

- prime ideals in the ring of integers R_k of the center k of the quaternion algebra A
- inequivalent primes elements $\mathfrak{p}$ in R_k generating these prime ideals (without loss of generality we will suppose $\mathfrak{p} > 0$)
- inequivalent discrete valuations v of A
- inequivalent completions $A_v = A_{\mathfrak{p}}, \mathcal{M}_v = \mathcal{M}_{\mathfrak{p}}$ of the quaternion algebra and a maximal order with respect to v

Recall that A_v is a skew field if and only if $\mathfrak{p}$ ramifies in A, i.e. if it belongs to the finite number of discriminant divisors. In this case, $\mathcal{M}_v$ is the unique maximal order of A_v, therefore there are no Eichler orders at all. In all other (unramified) cases we get matrix algebras $A_v \cong M_2(k_v)$, with maximal order $\mathcal{M}_v \cong M_2(R_v)$ where R_v denotes the ring of integers in the local field k_v, i.e. the completion of R_k in k_v. This ring has the unique prime ideal $\mathcal{P} = \mathfrak{p}R_v$, and all Eichler orders are conjugate to a ring of matrices

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{with} \quad a,b,d \in R_v \ , \ c \in \mathcal{P}^n \right\}$$

for some positive integer n ($\mathcal{P}^n$ is the *level* of the Eichler order). This local Eichler order is in fact an intersection $\mathcal{M}_v \cap \alpha^{-1}\mathcal{M}_v\alpha$ of two maximal orders conjugate by some $\alpha \in \mathcal{M}_v^* \left(\begin{smallmatrix} \mathfrak{p} & 0 \\ 0 & 1 \end{smallmatrix}\right) \subset M_2(R_v)$.

The study of the local situation will be crucial to get global consequences using global-local arguments.

2.2. The local situation

Suppose that the triangle group Δ is the norm 1 group of a maximal order $\mathcal{M}$, and that the surface group K is included in both Δ and $\alpha\Delta\alpha^{-1}$ for some $\alpha \in \mathrm{PSL}(2,\mathbb{R})$, so that K is included in the norm 1 group of the Eichler order $\mathcal{E} = \mathcal{M} \cap \alpha\mathcal{M}\alpha^{-1}$. The local situation is the following:

- For all valuations $v \in \mathrm{Ram}_f(A)$ the localised algebra A_v contains a unique maximal order, and therefore $\mathcal{E}_v = \mathcal{M}_v = \alpha\mathcal{M}_v\alpha^{-1}$;
- There is a finite number of $v \notin \mathrm{Ram}_f(A)$ such that $\mathcal{M}_v \neq \alpha\mathcal{M}_v\alpha^{-1}$.

Local Eichler orders in $M_2(k_v)$ are easy to study thanks to the tree structure of the maximal orders, mentioned in section 0.7. Recall that, if the valuation v corresponds to a prime ideal $\mathfrak{p}$, vertices in the tree correspond to maximal orders and two vertices are joined by an edge if and only if the corresponding maximal orders are conjugate under an element whose norm is in $R_k^*\mathfrak{p}$. The chain of inclusions

$$\mathcal{M}_v > \mathcal{M}_v \cap \left(\begin{smallmatrix} \mathfrak{p} & 0 \\ 0 & 1 \end{smallmatrix}\right)^{-1} \mathcal{M}_v \left(\begin{smallmatrix} \mathfrak{p} & 0 \\ 0 & 1 \end{smallmatrix}\right) > \ldots > \mathcal{M}_v \cap \left(\begin{smallmatrix} \mathfrak{p} & 0 \\ 0 & 1 \end{smallmatrix}\right)^{-n} \mathcal{M}_v \left(\begin{smallmatrix} \mathfrak{p} & 0 \\ 0 & 1 \end{smallmatrix}\right)^{n}$$

implies that a local Eichler order $\mathcal{E}_v = \mathcal{M}_v \cap \alpha\mathcal{M}_v\alpha^{-1}$ is contained in all the maximal orders corresponding to vertices lying in the path joining $\mathcal{M}_v$ and $\alpha\mathcal{M}_v\alpha^{-1}$. If $\mathcal{M}_v$ and $\alpha\mathcal{M}_v\alpha^{-1}$ are neighbours we will say that $\mathcal{E}$ is an Eichler order of level $\mathcal{P}$ and, more generally, if the path joining the two maximal orders has length n, we will say that $\mathcal{E}$ is an Eichler order of level $\mathcal{P}^n$.

We begin with the simplest case, which is local Eichler orders of level $\mathcal{P}$. Let q be the number of elements in the residue field $R_v/\mathcal{P} \cong \mathbb{F}_q$. We have the following result.

Lemma 2.1. *Let $\mathcal{M}_v = M_2(R_v)$ be a local maximal order in $A_v = M_2(k_v)$. The norm 1 group $\Phi_0 = \Phi_0(\mathcal{P})$ of an Eichler order $\mathcal{M}_v \cap \alpha\mathcal{M}_v\alpha^{-1}$ of level $\mathcal{P}$ has index $q+1$ in $\mathcal{M}_v^1$. Moreover, $\mathcal{M}_v$ and $\alpha\mathcal{M}_v\alpha^{-1}$ are the only maximal orders in which Φ_0 is contained.*

Proof. If one considers the canonical action of $\mathcal{M}_v^1 = M_2(R_v)$ on the projective line $\mathbb{P}^1(\mathbb{F}_q)$ given by reduction modulo $\mathcal{P}$, the groups Φ_0 correspond to the subgroups fixing one point. There are therefore $q+1$ of them, and this number coincides with the index. If the Eichler order was included in further maximal orders apart from $\mathcal{M}_v$ and $\alpha\mathcal{M}_v\alpha^{-1}$, it would correspond to a longer path in the tree of maximal orders, which is a contradiction since it has level $\mathcal{P}$. □

Let $\mathcal{O} \subset A$ be a maximal order in the quaternion $k-$algebra A and $v \notin \mathrm{Ram}(A)$ an unramified valuation of A corresponding to the prime $\mathfrak{p}$, such that $\mathcal{O}_v = M_2(R_v)$. Let $\mathcal{E}(\mathfrak{p})$ denote the local Eichler order $\mathcal{M}_v \cap \left(\begin{smallmatrix} \mathfrak{p} & 0 \\ 0 & 1 \end{smallmatrix}\right)^{-1} \mathcal{M}_v \left(\begin{smallmatrix} \mathfrak{p} & 0 \\ 0 & 1 \end{smallmatrix}\right)$, whose norm 1 group we have denoted by $\Phi_0(\mathcal{P})$. We will write $\Delta_0(\mathfrak{p})$ for the norm 1 group of the Eichler order $\mathcal{E}$ in A that corresponds via the bijection in Lemma 0.1 to the family

$$\{(\mathcal{E}_v) : \mathcal{E}_v = \mathcal{E}(\mathfrak{p}) \text{ if } v = \mathfrak{p}, \text{ and } \mathcal{E}_v = \mathcal{O}_\mathfrak{p}, \text{ if } v \neq \mathfrak{p}\}.$$

In the particular case where $A = M_2(k)$ and R_k is a principal ideal domain all maximal orders are conjugate, and we can suppose that $\mathcal{O} = M_2(R_k)$. Therefore $\Delta_0(\mathfrak{p})$ coincides with the congruence subgroup

$$\Delta_0(\mathfrak{p}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Delta \subset M_2(R_k) : c \equiv 0 \bmod \mathfrak{p} \right\}.$$

We have the following necessary condition for the existence of at least two different uniform dessins of the same type on a Riemann surface of genus > 1. This result will be crucial for the construction of low genus examples.

Theorem 2.3. *Let K be an arithmetic Fuchsian surface group contained in the triangle group $\Delta = \Delta(l, m, n)$, and suppose that Δ is the norm 1 group $\mathcal{M}^1$ in a maximal order $\mathcal{M}$ of a quaternion algebra A defined over the totally real field k with ring of integers R_k. The group K is contained in more than one triangle group of type (l, m, n) if and only if K is contained in a group conjugate in Δ to $\Delta_0(\mathfrak{p})$, where $\mathfrak{p}$ is a prime of k not dividing the discriminant of A.*

Proof. Suppose first that $K < \Delta \cap \alpha\Delta\alpha^{-1}$ for some $\alpha \in \mathrm{PSL}(2, \mathbb{R})$. We can suppose $\alpha \in A$ by the Skolem–Noether Theorem, and then by Lemma 0.1 there exists at least one valuation $\mathfrak{p} \notin \mathrm{Ram}(A)$ such that $\mathcal{M}_\mathfrak{p} \neq \alpha\mathcal{M}_\mathfrak{p}\alpha^{-1}$. For this valuation, we can suppose modulo conjugation that $\mathcal{M}_\mathfrak{p} \cap \alpha\mathcal{M}_\mathfrak{p}\alpha^{-1} \subset \mathcal{E}(\mathfrak{p})$, and therefore $K < \Delta \cap \alpha\Delta\alpha^{-1} < \Delta_0(\mathfrak{p})$.

The converse follows directly from the definitions. □

The following lemma describes the norm 1 groups of the intersections of local Eichler orders of level $\mathcal{P}$.

LEMMA 2.2. *Let $\mathcal{M}_v = M_2(R_v)$ be a local maximal order in $A_v = M_2(k_v)$. Now we consider $\mathcal{M}_v^1$ and its subgroups as subgroups of* $\mathrm{PSL}(2, R_v)$*, i.e. modulo* $\pm\mathrm{Id}$*. Then*

(i) *The norm* 1 *group $\Phi_0^0 = \Phi_0^0(\mathcal{P})$ of the intersection of two Eichler orders of level $\mathcal{P}$ has index $q(q+1)$ in $\mathcal{M}_v^1$. Moreover, Φ_0^0 is contained in 3 different maximal orders if $q > 3$, 5 if $q = 3$ and 4 if $q = 2$.*

(ii) *The norm* 1 *group $\Phi(\mathcal{P})$ of the intersection of more than two Eichler orders of level $\mathcal{P}$ is the principal congruence subgroup modulo $\mathcal{P}$ of $\mathcal{M}_v^1$, a normal subgroup of $\mathcal{M}_v^1$ of index $\frac{1}{2}q(q^2-1)$ (omit the denominator* 2 *if q is a $2-$power). It is the intersection of all such Eichler orders of level $\mathcal{P}$ and is included in $q+2$ different maximal orders.*

PROOF. If we consider again the canonical operation of $\mathcal{M}_v^1$ on the projective line $\mathbb{P}^1(\mathbb{F}_q)$, the groups Φ_0^0 correspond to the elements fixing two points. If more than two points are fixed, automatically all points of the projective line are fixed, hence the case in (*ii*) already gives the principal congruence subgroup.

The cases $q = 2$ and 3 play a special role because for them $\Phi_0^0(\mathcal{P}) = \Phi(\mathcal{P})$: recall that we see them as projective groups, and since the determinants are 1, in the case of small q all matrices in $\Phi_0^0(\mathcal{P})$ are congruent $\bmod \mathcal{P}$ to $\pm$ the unit matrix.

For the calculation of the indices one may consult [65] p. 109 or mimic a proof from any book about modular forms. Alternatively one may consider the groups involved as the stabilizers of one point, two points or the whole projective line, and then the index is given by the number of elements in the orbit of the fixed points. □

In a similar way to the case of $\Delta_0(\mathfrak{p})$ and $\Phi_0(\mathcal{P})$, we can define the principal congruence subgroup $\Delta(\mathfrak{p})$ as the subgroup of Δ whose localisation in $\mathfrak{p}$ coincides with $\Phi(\mathcal{P})$. The existence and uniqueness of such a subgroup is granted by the Strong Approximation Theorem (see for example [65] or [51]), which is an extreme version of the Chinese remainder theorem for certain matrix groups and whose formal statement exceeds the purposes of this thesis. In the particular case where $A = M_2(k)$ we can again suppose that $\mathcal{O} = M_2(R_k)$ and therefore

$$\Delta(\mathfrak{p}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod \mathfrak{p} \right\}.$$

LEMMA 2.3. *For integers $n > 1$ there are $q^{n-1}(q+1)$ different local Eichler orders $\mathcal{M}_v \cap \alpha^{-1}\mathcal{M}_v\alpha$ of level $\mathcal{P}^n$. Their norm* 1 *groups $\Phi_0(\mathcal{P}^n)$ have index $q^{n-1}(q+1)$ in $\mathcal{M}_v^1$. The intersection of all these norm* 1 *groups is the principal congruence subgroup $\Phi(\mathcal{P}^n)$, which is included in $\frac{(q+1)(q^n-1)}{q-1} + 1$ different maximal orders.*

PROOF. To prove that there are precisely $q^{n-1}(q+1)$ such Eichler orders of level $\mathcal{P}^n$ with norm 1 group $\Phi_0(\mathcal{P}^n)$ one may just count paths of length n in the tree of maximal orders, with one end fixed in the vertex $\mathcal{M}_v$. For the index formula one may use the same argument of the previous Lemma, this time defining an action of $\mathcal{M}_v^1$ on the "fake projective line" $\mathbb{P}_m^1$ over the residue class ring $R_k/\mathfrak{p}^m \cong R_v/\mathcal{P}^m$, which is the set of pairs of residue classes, not both in $\mathfrak{p}R_k/\mathfrak{p}^m$, modulo the unit group of this residue class ring (see also [65] p. 55).

The intersection of all Eichler orders of level $\mathcal{P}^n$ is then included in all the maximal orders at distance n from $\mathcal{M}_v$. □

As an illustration for the result concerning the principal congruence subgroups, we show in Figure 2.1 the picture of the subtree for $\Phi(\mathfrak{p}^2)$ in the case $q = 7$. We can define global principal congruence subgroups $\Delta(\mathfrak{p}^n)$ of higher level in the same way as above.

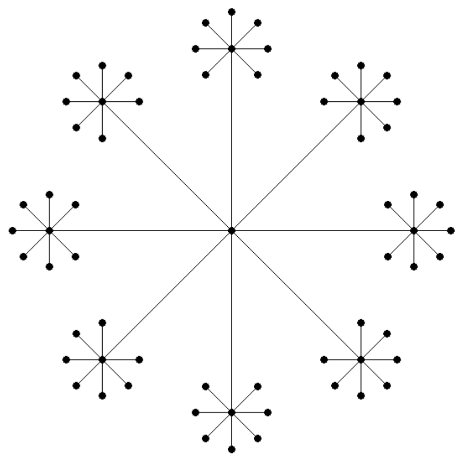


FIGURE 2.1. Subtree for $\Phi(\mathfrak{p}^2)$ in the local algebra $A_\mathfrak{p}$ for $q = 7$

2.3. Global consequences

To understand Theorem 2.3 and construct examples in low genus, suppose that $K < \Delta_0(\mathfrak{p})$ for some $\mathfrak{p} \notin \mathrm{Ram}(A)$. This group $\Delta_0(\mathfrak{p})$ is always contained in the so called *Fricke extension* $\Delta_0^{\mathrm{Fr}}(\mathfrak{p})$ which, in the case of $A = M_2(k)$, is the index two extension of $\Delta_0(\mathfrak{p})$ by the element

$$\alpha = \frac{1}{\sqrt{\mathfrak{p}}}\begin{pmatrix} 0 & \mathfrak{p} \\ -1 & 0 \end{pmatrix} \in \mathrm{PSL}(2, \mathbb{R}),$$

where $\mathfrak{p}$ is chosen to be totally positive. This element clearly normalises $\Delta_0(\mathfrak{p})$, but not Δ. The action induced by conjugation on $\Delta_0(\mathfrak{p})$ is called the *Fricke involution*. As a consequence the group $K < \Delta_0(\mathfrak{p})$ is included in both Δ and $\alpha\Delta\alpha^{-1}$, yielding two different uniform dessins in $\mathbb{H}/K$. In the ramified case, the Fricke involution can be seen in the localised algebra $A_\mathfrak{p}$ as the element $\left(\begin{smallmatrix} 0 & 1/\mathfrak{p} \\ -1 & 0 \end{smallmatrix}\right)$, which interchanges by conjugation $\mathcal{M}_v$ and $\left(\begin{smallmatrix} \mathfrak{p} & 0 \\ 0 & 1 \end{smallmatrix}\right)^{-1}\mathcal{M}_v\left(\begin{smallmatrix} \mathfrak{p} & 0 \\ 0 & 1 \end{smallmatrix}\right)$, and therefore fixes $\mathcal{E}(\mathfrak{p})$.

Now we will concentrate on a series of striking examples. Take Δ of signature $(2, 3, 7)$. According to [64] this is the norm 1 group of a maximal order $\mathcal{M}$ in a quaternion algebra A over the cubic field $k = \mathbb{Q}(\cos\frac{2\pi}{7})$.

It is well known that Hurwitz curves are uniformised by normal subgroups K of the triangle group $\Delta(2,3,7)$ and that, in particular, one has $\mathrm{Aut}(S) \cong \Delta(2,3,7)/K$. A classical theorem by Macbeath ([50]) shows that $\mathrm{PSL}(2,\mathbb{F}_q)$ is a Hurwitz group exactly in the following cases

(i) $q = 7$,
(ii) $q = p$ prime for $p \equiv \pm 1 \bmod 7$,
(iii) $q = p^3$ for p prime and $p \equiv \pm 2$ or $\pm 3 \bmod 7$.

Accordingly, the corresponding Riemann surfaces are usually known as *Macbeath–Hurwitz curves.*

It was proved in [19] by A. Dzămbić that all Macbeath-Hurwitz curves can be constructed arithmetically as follows. The triangle group $\Delta(2,3,7)$ is the norm 1 group of a maximal order in the quaternion A over the field $k = \mathbb{Q}(\cos \pi/7)$ which is ramified exactly over the two non-trivial Archimedean valuations of k. Any rational prime p defines an ideal $p\mathcal{R}_k$ in $\mathcal{R}_k$ such that

(i) if $p = 7$ then p is ramified and $p\mathcal{R}_k = \mathfrak{p}^3$ for a prime ideal $\mathfrak{p} \subset \mathcal{R}_k$ of norm $q = N(\mathfrak{p}) = 7$;
(ii) if $p \equiv \pm 1 \bmod 7$ then p splits, i.e. $p\mathcal{R}_k = \mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_3$ for prime ideals $\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3 \subset R_k$ of norm $q = N(\mathfrak{p}_i) = p$;
(iii) if $p \equiv \pm 2$ or $\pm 3 \bmod 7$ then p is inert, i.e. $p\mathcal{R}_k$ is a prime ideal in $\mathcal{R}_k$ of norm $q = N(\mathfrak{p}) = p^3$.

For every prime $\mathfrak{p}$ in $\mathcal{R}_k$ we can define the subgroup of matrices of $\Delta(2,3,7)$ congruent to the identity modulo $\mathfrak{p}$. This is a normal torsion-free subgroup of $\Delta(2,3,7)$ with quotient group isomorphic to $\mathrm{PSL}(2,\mathbb{F}_q)$ where $q = N(\mathfrak{p})$, yielding therefore a Macbeath-Hurwitz curve.

The first cases are:

- Klein's quartic. Its surface group is $\Delta(\mathfrak{p})$ for a prime $\mathfrak{p}$ dividing 7, ramified of order 3 and of residue degree 1 in the extension $\mathbb{Q}(\cos\frac{2\pi}{7})/\mathbb{Q}$. With $q = 7$ we see that Klein's quartic has 8 conjugate uniform dessins of type $(2,3,7)$ plus the usual regular one.
- Macbeath's curve of genus 7 with automorphism group $\mathrm{PSL}(2,\mathbb{F}_8)$ has the surface group $\Delta(2)$ for the prime $\mathfrak{p} = 2$, inert and of residue degree 3 in the extension $\mathbb{Q}(\cos\frac{2\pi}{7})/\mathbb{Q}$. With $q = 8$ one has 9 uniform dessins plus a regular one on the curve.
- Three non-isomorphic curves in genus 14 whose automorphism groups are isomorphic to $\mathrm{PSL}(2,\mathbb{F}_{13})$ and whose surface groups are the principal congruence subgroups $\Delta(\mathfrak{p}_j), j = 1,2,3$ for the (completely decomposed) primes $\mathfrak{p}_j$ dividing 13. Their residue degree is 1, hence one has $q+1 = 14$ uniform dessins of type $(2,3,7)$ on each curve plus a regular one.

All dessins mentioned here are clearly not renormalisations of each other since the signature consists of three different entries. On the other hand, in all these cases we have one regular dessin and $q+1$ uniform non-regular ones which form an orbit under the automorphism group of the curve: the $q+1$ norm 1 groups of type $\Delta_0(\mathfrak{p})$ are conjugate under the action of Δ (in other words, the $q+1$ Eichler orders of level $\mathcal{P}$ form a $\Delta-$invariant set), so these dessins are equivalent under automorphisms of the curve.

One can consider the growth of the maximal number of uniform dessins on surfaces $\mathbb{H}/K$, as a function of the index $[\Delta : K]$ in a given triangle group. Global-local arguments yield the following bound.

THEOREM 2.4. *Let the Fuchsian group Δ be the norm 1 group of a maximal order in a quaternion algebra. For each positive integer $m > 0$, the maximum number of conjugates of Δ in which any Fuchsian group $K < \Delta$ of index at most can be included is $O(\sqrt[3]{m})$ and this upper bound is optimal in the following sense. There are sequences of surface groups $K_n < \Delta$ with indices $[\Delta : K_n] \to \infty$ such that, if we write d_n for the number of all residue classes $\alpha \in \mathrm{PSL}(2, \mathbb{R})/N(\Delta)$ with the property $K_n \subset \alpha\Delta\alpha^{-1}$, we have*

$$\lim_{n\to\infty} \frac{d_n}{\sqrt[3]{2[\Delta : K_n]}} = 1.$$

The proof of this result follows from considering local bounds and applying a local-global argument based on the Strong Approximation Theorem. For the sequence K_n one may take any sequence of principal congruence subgroups $\Delta(\mathfrak{p})$ with prime ideals $\mathcal{P} = \mathfrak{p}R_v$ such that $R_k/\mathfrak{p} \cong \mathbb{F}_q$, with $q \to \infty$. Observe that only finitely many among the K_n can have torsion.

However, in these examples we have only the rather modest number of two essentially different (non-isomorphic) dessins of the same type. Nevertheless, replacing these congruence groups with subgroups of small index we can remove automorphisms such that most of the uniform dessins found here become inequivalent. As a consequence, and describing the growth result given in Theorem 2.4 in terms of the genus, we get the following corollary.

COROLLARY 2.1. *The number of uniform dessins not equivalent under renormalisation or automorphisms on a Belyi surface grows with the genus g at most as a multiple of $\sqrt[3]{g}$, and this bound is optimal.*

We refer to [**34**] for full details of the proofs.

2.4. A geometrical description

We explore now the examples given in section 2.3 in a more geometrical way.

2.4.1. Klein's quartic. Klein's quartic is a genus three surface uniformised by a group K generated by certain side-pairings in the regular 14-gon P with angle $2\pi/7$ (see Figure 2.2). The (black and white) triangles in Klein's original picture are related to the triangle group $\Delta(2, 3, 7)$ of signature $(2, 3, 7)$ in which K is normally contained with index 168.

The inclusion $K \lhd \Delta(2, 3, 7)$ induces a regular Belyi function on K. The corresponding regular dessin $\mathcal{D}$ can be easily depicted in P with the help of the triangle tessellation associated to $\Delta(2, 3, 7)$ (see left picture on Figure 2.3).

Now rotate $\mathcal{D}$, or rather its lift to the universal covering $\mathbb{D}$, by an angle $2\pi/14$ around the origin. The graph $\mathcal{D}'$ obtained is compatible with the side-pairing identifications, hence it is a well defined dessin on the surface. It is rather obvious that $\mathcal{D}'$ decomposes the surface into 24 heptagons in the same way as $\mathcal{D}$ does. In other words $\mathcal{D}'$ is also a uniform $(2, 3, 7)$ dessin on $\mathbb{H}/K$ (see right picture on Figure 2.3). Note that the rotation that transforms $\mathcal{D}$ into $\mathcal{D}'$ does not correspond to any automorphism of the surface, and in fact both dessins are not isomorphic

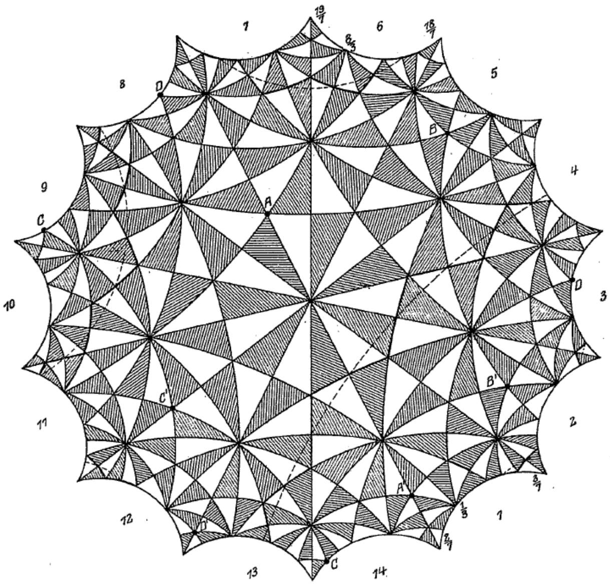


FIGURE 2.2. Klein's surface is obtained by the side pairing $1 \leftrightarrow 6$, $3 \leftrightarrow 8$, $5 \leftrightarrow 10$, $7 \leftrightarrow 12$, $9 \leftrightarrow 14$, $11 \leftrightarrow 2$, $13 \leftrightarrow 4$.

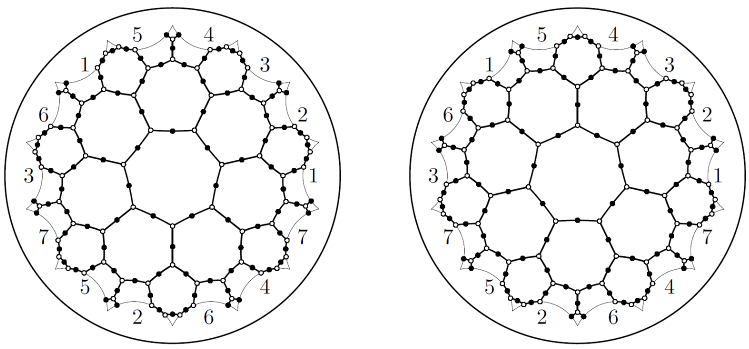
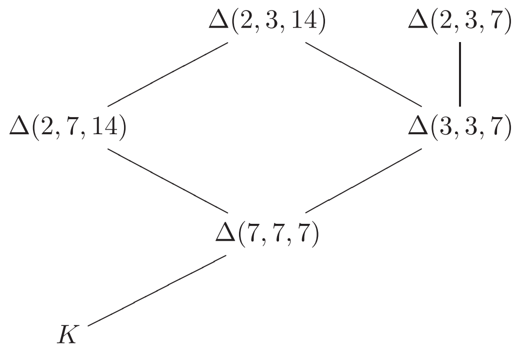


FIGURE 2.3. Klein's regular $(2,3,7)$ dessin $\mathcal{D}$ and a uniform one $\mathcal{D}'$.

since $\mathcal{D}'$ is not regular (it can be checked that the automorphism group $\mathrm{Aut}(S)$ does not act transitively on the edges of this new dessin).

This existence of a new uniform dessin of type $(2,3,7)$ is clear if one studies all triangle groups in which K is contained. The group K corresponds to $\Delta(\mathfrak{p})$, for a prime $\mathfrak{p}$ dividing 7 in $\mathbb{Q}(\cos\pi/7)$. The surface group K is a normal subgroup of $\Delta(2,3,7)$, but it is also contained normally in the group $\Delta(7,7,7)$ that has one seventh of the 14-gon as fundamental domain. The corresponding regular $(7,7,7)-$dessin lies in the border of the polygon: it has one black vertex, one white vertex, and seven edges. There is even a group $\Delta(3,3,7)$ lying between $\Delta(7,7,7)$ and $\Delta(2,3,7)$ that defines another regular dessin of type $(3,3,7)$. The chain of inclusions $K < \Delta(7,7,7) < \Delta(3,3,7) < \Delta(2,3,7)$ means that the corresponding regular dessins are related by refinement. Moreover, it can be checked that this group $\Delta(3,3,7)$ corresponds to $\Delta_0(\mathfrak{p})$, and therefore one has an index two extension $\Delta_0^{\mathrm{Fr}}(\mathfrak{p}) = \Delta(2,3,14)$. In fact, the full diagram of triangle groups lying above K can be found looking at Singerman's inclusion list:

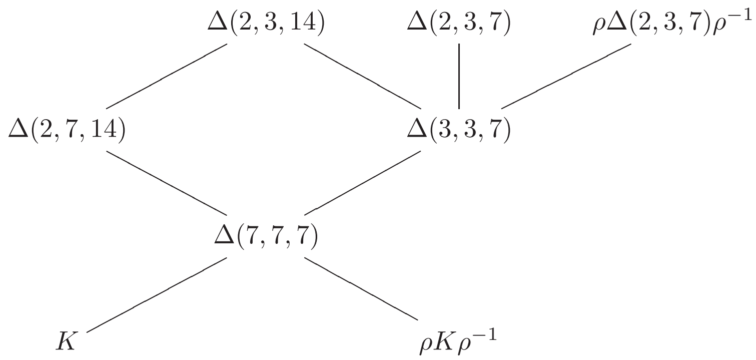
(2.1)



The groups $\Delta(2,7,14)$ and $\Delta(2,3,14)$ are the index two (therefore normal) extensions of $\Delta(7,7,7)$ and $\Delta(3,3,7)$ obtained by addition of a new element ρ (which induces the Fricke involution of $\Delta_0(\mathfrak{p})$) which is a rotation of angle $2\pi/14$ around the origin. The corresponding dessins of type $(2,7,14)$ and $(2,3,14)$ are not regular but only uniform (as already noticed in [**62**]), and are obtained from those of types $(7,7,7)$ and $(3,3,7)$ by colouring all the vertices with the same colour, say black, and then adding white vertices at the midpoints of the edges.

Conjugation of diagram (2.1) by ρ fixes all the groups except K and $\Delta(2,3,7)$:

(2.2)



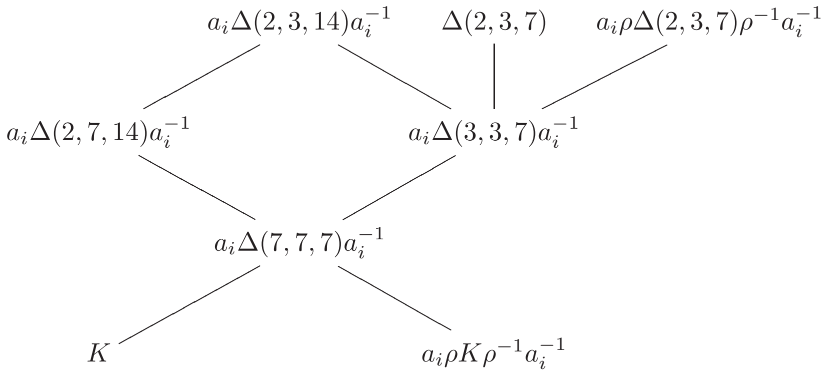
The inclusion $K < \rho\Delta(2,3,7)\rho^{-1}$ corresponds to the uniform dessin $\mathcal{D}'$ described above. Since the normaliser of K is $\Delta(2,3,7)$ the inclusion of K in the triangle group $\rho\Delta(2,3,7)\rho^{-1}$ is not normal, hence $\mathcal{D}'$ is not regular.

Now we focus in the group $\Delta(3,3,7)$ lying in the middle of diagrams (2.1) and (2.2). It is a known fact ([**35**]) that a given triangle group of type $(3,3,7)$ is contained in precisely two different groups of signature $(2,3,7)$, i.e. $\Delta(2,3,7)$ and $\rho\Delta(2,3,7)\rho^{-1}$ in our case. Conversely, any given $\Delta(2,3,7)$ contains eight different subgroups of signature $(3,3,7)$, all conjugate in $\Delta(2,3,7)$. From the point of view of local quaternion algebras, this is a consequence of Lemmas 2.1 and 2.2.

Let $a_0\Delta(3,3,7)a_0^{-1} = \Delta(3,3,7), a_1\Delta(3,3,7)a_1^{-1}, \ldots, a_7\Delta(3,3,7)a_7^{-1}$ be the 8 subgroups of $\Delta(2,3,7)$ conjugate to $\Delta(3,3,7)$, with $a_i \in \Delta(2,3,7)$.

If we conjugate diagram (2.2) by a_i we get

(2.3)



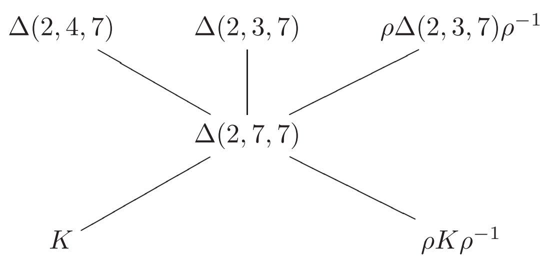
Note that only $\Delta(2,3,7)$ and K remain fixed by this conjugation, since a_i belongs to $\Delta(2,3,7)$, the normaliser of K.

The inclusion $K < a_i\rho\Delta(2,3,7)\rho^{-1}a_i^{-1}$ induces a new uniform (but not regular) dessin of type $(2,3,7)$ on $\mathbb{H}/K$. It is related to the uniform dessin $\mathcal{D}'$ by the automorphism induced by a_i, and to the regular dessin $\mathcal{D}$ by a hyperbolic rotation of angle $2\pi/14$ around the center of certain face of $\mathcal{D}$.

2.4.2. Macbeath's curve of genus seven. The description of the uniform $(2,3,7)$ dessins on Macbeath curve goes more or less along the same lines as in the case of Klein's quartic. Again the surface group K is included normally in $\Delta(2,3,7)$. The role played by the group $\Delta(3,3,7)$ in Klein's quartic is played here by $\Delta(2,7,7)$, which this time corresponds to $\Delta_0(2)$. Note that the inclusion $\Delta(2,7,7) < \Delta(2,3,7)$ is also very special (cf. [**35**]). The number of conjugate subgroups of type $(2,7,7)$ inside $\Delta(2,3,7)$ is nine, and any given $\Delta(2,7,7)$ is contained in two different groups of type $(2,3,7)$ (this is again consequence of Lemmas 2.1 and 2.2. The normaliser of $\Delta(2,7,7)$ is now a $(2,4,7)$–group obtained by adding a rotation ρ of order 4 around any of the points of order 2 in $\Delta(2,7,7)$.

This new element does not normalise $\Delta(2,3,7)$, so conjugation by ρ gives rise to the second group $\rho\Delta(2,3,7)\rho^{-1}$ in which $\Delta(2,7,7)$ is included:

(2.4)



The inclusion of K inside $\Delta(2,3,7)$ and $\rho\Delta(2,3,7)\rho^{-1}$ determines two non isomorphic dessins on Macbeath's curve. Once more the second inclusion is not normal, and accordingly the second dessin is uniform but not regular.

We can proceed in the same way with the other eight $(2,7,7)$–groups contained inside $\Delta(2,3,7)$ to get diagrams similar to diagram (2.3). This way we find the nine

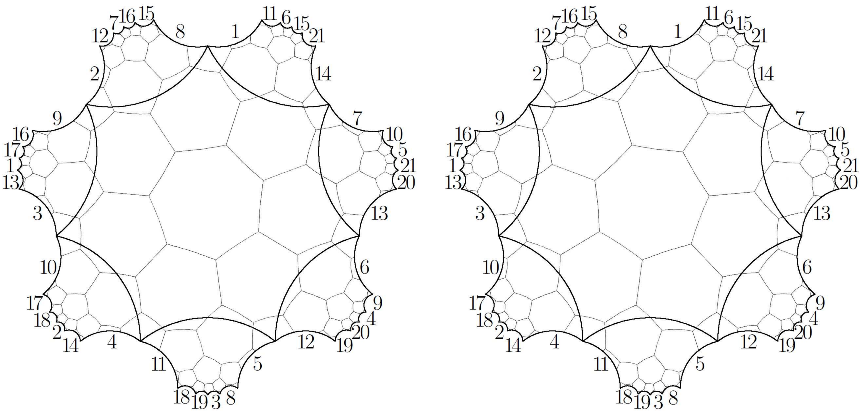


FIGURE 2.4. Face decomposition associated to regular and uniform dessins of type $(2,3,7)$ on Macbeath's surface.

(isomorphic) uniform dessins predicted by the arithmetic arguments of Section 2.3. There is obviously as well a uniform dessin of type $(2,4,7)$, as already noticed in [62].

2.4.3. Macbeath-Hurwitz curves of genus 14. The third example given in section 2.3 arises from the consideration of the three (torsion free) groups $K_i = \Delta(\mathfrak{p}_i) \lhd \Delta(2,3,7)$ for inequivalent primes $\mathfrak{p}_1$, $\mathfrak{p}_2$ and $\mathfrak{p}_3$ dividing 13 in $\mathbb{Q}(\cos\frac{\pi}{7})$. These groups correspond to three Galois conjugate curves of genus 14 with a regular $(2,3,7)$ dessin ([63]).

Now for each of these primes, we find $\Delta_0(\mathfrak{p}_i)$ lying between $\Delta(\mathfrak{p}_i)$ and $\Delta(2,3,7)$. Its index inside $\Delta(2,3,7)$ is 14. By Singerman's method for the determination of signatures of subgroups of Fuchsian groups ([60]) it can be seen that $\Delta_0(\mathfrak{p}_i)$ is a group of signature $\langle 0; 2, 2, 3, 3\rangle$.

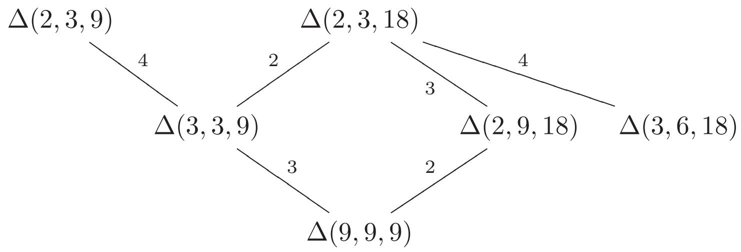
There is again an element ρ_i in the normaliser of $\Delta_0(\mathfrak{p}_i)$ that conjugates $\Delta(2,3,7)$ into a different group. The inclusion of $\Delta(\mathfrak{p}_i)$ inside $\rho_i\Delta(2,3,7)\rho_i^{-1}$ is no longer normal and gives rise to a non-regular uniform dessin on the same Riemann surface.

Moreover, $\Delta(2,3,7)$ contains 14 different subgroups conjugate to $\Delta_0(\mathfrak{p}_i)$. All of them include $\Delta(\mathfrak{p}_i)$, therefore arguing as above we find 14 isomorphic uniform $(2,3,7)$ dessins.

2.5. Uniform dessins in genus 2 surfaces

A complete list of all (isomorphism classes of) uniform dessins in genus 2 is given in [62]. It is however not obvious if and when two such dessins may belong to the same surface.

Let us focus on the triangle group $\Delta = \Delta(2,3,9)$, which corresponds to the norm 1 group of a maximal order in a quaternion algebra and which contains the triangle group $\Delta(3,3,9)$ with index 4. We depict here the diagram of inclusions appearing in [64]:



The associated quaternion algebra is defined over the cubic field $k := \mathbb{Q}(\cos\frac{\pi}{9})$ and is unramified. In k we have a ramified prime $\mathfrak{p} \mid 3$ of norm 3 such that $\Delta_0(\mathfrak{p}) = \Delta(3,3,9)$. There has to be an extension of index 2 by the Fricke involution, and in fact $\Delta(3,3,9)$ is normalised by the triangle group $\Delta(2,3,18)$.

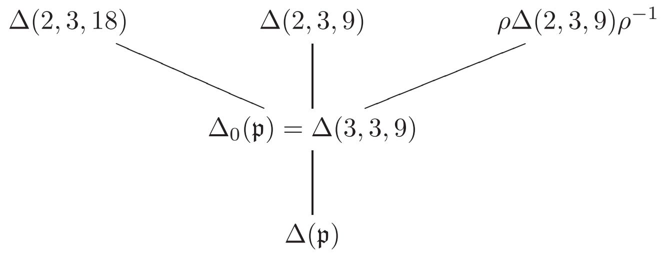
Therefore, we have the following chain of inclusions:

$$\Delta \overset{4}{>} \Delta_0(\mathfrak{p}) \overset{3}{>} \Delta(\mathfrak{p}),$$

where $\Delta_0(\mathfrak{p}) = \Delta(3,3,9)$ and $\Delta(\mathfrak{p})$ is the principal congruence subgroup of level $\mathfrak{p}$, a Fuchsian group of signature $\langle 0; 3,3,3,3\rangle$. Moreover, since $q = 3$ we have $\Delta(\mathfrak{p}) \simeq \Delta_0(\mathfrak{p}^2)$.

By the arithmetic theory developed before, the Fricke involution conjugates $\Delta(2,3,9)$ into another group $\rho\Delta(2,3,9)\rho^{-1}$ such that:

(2.5)



Once again ρ is an extra rotation in $\Delta(2,3,18) = N(\Delta(3,3,9))$ of order 2 around a fixed point of order 9. Let us note that conjugation by the Fricke involution gives precisely the isomorphism between $\Delta(\mathfrak{p})$ and $\Delta_0(\mathfrak{p}^2) = \rho\Delta(\mathfrak{p})\rho^{-1}$.

Now by Theorem 2.3 every surface group inside $\Delta(3,3,9)$ will have at least two $(2,3,9)$ dessins. By the list in [62] we know that in genus 2 there are 4 different dessins of this type. For two of them it can be seen, by computing the monodromies and constructing a fundamental domain, that the Fricke involution is an automorphism of the surface, and so the two dessins arising from the arithmetic construction are isomorphic (see also [31]).

The other two are the dual dessins considered in [62], Section 11(d). To find its surface group we can follow once more Singerman's procedure, and it can be seen that it is possible to find a (normal) torsion free subgroup K of index 3 in $\Delta(\mathfrak{p})$. The indices $[\Delta(3,3,9) : K] = 9$ and $[\Delta(2,3,9) : K] = 36$ tell us that it corresponds indeed to a genus 2 surface.

One can compute the monodromies of the two dessins induced by $\Delta(2,3,9)$ and $\rho\Delta(2,3,9)\rho^{-1}$ by computer means to check that they are non-conjugate inside $\mathfrak{S}_{36}$ so they are not isomorphic as we already knew. They are not equivalent either under automorphisms or renormalisation.

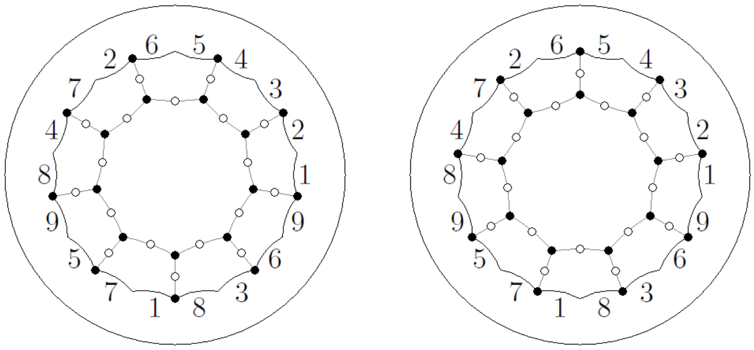
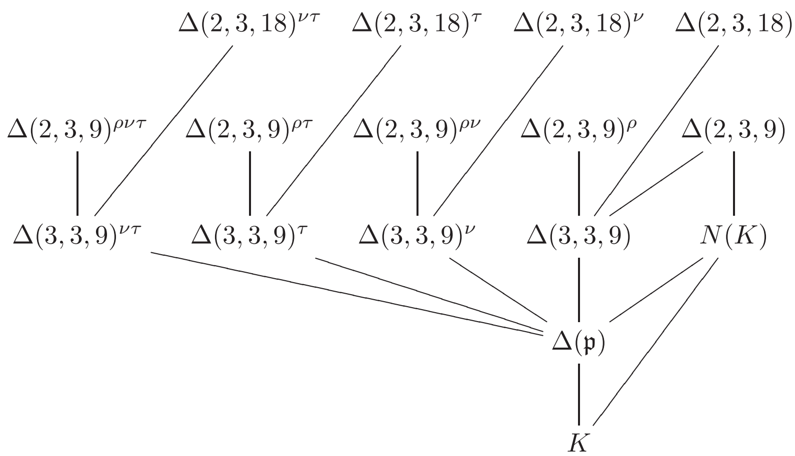


FIGURE 2.5. Two non isomorphic uniform dessins of type $(2, 3, 9)$ in the same surface.

By [31] we know that the automorphism group of this surface $\mathrm{Aut}(\mathbb{H}/K) \simeq N(S)/S$ is generated by the hyperelliptic involution ν and two automorphisms τ and σ_3 of order 2 and 3 respectively. From the arithmetic point of view, we can even say that $\langle K, \widetilde{\sigma}_3\rangle \simeq \Delta(\mathfrak{p})$, where $\widetilde{\sigma}_3$ denotes the lift of σ_3 to $\mathbb{H}$.
The lifts of all these automorphisms lie inside $\Delta(2, 3, 9)$, but $\widetilde{\nu}, \widetilde{\tau}, \widetilde{\nu\tau} \in N(K)$ do not belong to $\rho\Delta(2, 3, 9)\rho^{-1}$. Conjugation of $\rho\Delta(2, 3, 9)\rho^{-1}$ by each of these elements will determine another $(2, 3, 9)$-dessin isomorphic to the second one.
The same can be applied to $\Delta(3, 3, 9)$ and $\Delta(2, 3, 18)$. In particular $\Delta(3, 3, 9)$, $\widetilde{\nu}(\Delta(3, 3, 9))\widetilde{\nu}^{-1}$, $\widetilde{\tau}(\Delta(3, 3, 9))\widetilde{\tau}$ and $\widetilde{\nu\tau}(\Delta(3, 3, 9))(\widetilde{\nu\tau})^{-1}$ are the 4 different $(3, 3, 9)$ groups lying below a given $\Delta(2, 3, 9)$ ([35], p.9, Thm.6).

The following diagram of inclusions shows all the dessins (modulo renormalisation) in this surface. The notation G^σ stands for conjugation by σ:

(2.6)



To sum up, there are (up to renormalisation) four different $(3, 3, 9)$ dessins on the surface $\mathbb{H}/K$ studied here, forming one orbit under the automorphism group $N(K)/K \simeq D_3 \times C_2$ acting on the edges of the subtree given in Figure 2.6. On the other hand one has four $(2, 3, 9)$ dessins equivalent under the automorphism group

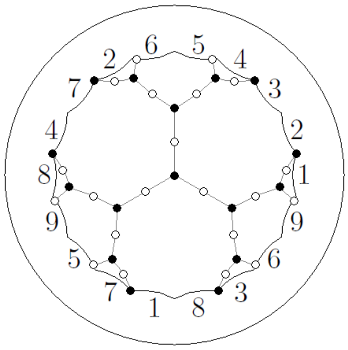


FIGURE 2.6. Subtree $\mathcal{T}(\Delta_{\mathfrak{p}}(\mathfrak{p}))$ and the image of the second dessin under the hyperelliptic involution ν.

plus one stabilized by $N(K)/K$, corresponding to the mid-vertex of the subtree, not isomorphic to the others.

REMARK 2.1. According to [62] an equation for $\mathbb{H}/K$ is $y^3 = (x-1)(x^3-1)$. We have found that $y^2 = x^6 + 8x^3 + 4$ is a hyperelliptic model of this surface.

In low genus there is another theoretical approach to the problem of different uniform dessins living on the same surface. This method is based on studying the location in a fundamental domain of the surface of the possible face-centres of different dessins, and the tools used are mainly from hyperbolic geometry. Proceeding in this way we are able to classify all unicellular (i.e. with only one face) uniform dessins in genus 2. We sketch here the ideas behind the methodology.

The hyperbolic distances between the fixed points of any two elements in the conjugacy class of z inside $\Delta(l,m,n) = \langle x, y, z : x^l = y^m = z^n = xyz = 1\rangle$ form a discrete set $d(l,m,n) = \{d_1 < d_2 < \cdots\}$ that does not depend on the choice of the triangle group within its conjugacy class. We will call $d(l,m,n)$ the set of *admissible distances* for the type (l,m,n). These hyperbolic distances were already used in [9] and [31]. The point is now that, if a uniform dessin $\mathcal{D}$ lives on a surface $S = \mathbb{D}/K$, which is given as a fundamental polygon in $\mathbb{D}$ with identified edges, and a point $\mathrm{w} \in \mathbb{D}$ corresponds to a face center of some other dessin $\mathcal{D}'$, then for every $\gamma \in K$, one has $d_{\mathbb{D}}(\mathrm{w}, \gamma(\mathrm{w})) \in d(l,m,n)$, where $d_{\mathbb{D}}$ stands for the hyperbolic distance on the disc. On the other hand, given $K < \Delta$, any point of $\mathbb{D}$ moved an admissible distance by every transformation in K will be called an *admissible point*, since it is a candidate for being the face center of a uniform dessin of type (l,m,n) on the surface $\mathbb{D}/K$. Since face centers of uniform (l,m,n)-dessins on S must be detectable as admissible points, we look for points in $\mathcal{P}$ that are moved an admissible distance by (at least) all the side-pairings generating K, and this search is computer aided. Discerning the true face centers among these admissible points requires further arguments, mainly on the automorphisms of S.

The results are presented mainly through pictures like the one in Figure 2.7. The edges and the vertices z_i of the fundamental polygon $\mathcal{P}$ are labelled counterclockwise. The i-th edge joins z_{i-1} and z_i, and the edge 1 is the one intersecting $\mathbb{R}^+$. The hyperbolic midpoint of the i-th edge is denoted by p_i, and $F^-_{(i,j)}$ and $F^+_{(i,j)}$

denote the repelling and attracting fixed points of $\gamma_{(i,j)}$, the transformation that sends the i-th edge of $\mathcal{P}$ to the j-th one. The arcs joining these two points represent admissible arcs for $\gamma_{(i,j)}$, i.e. arcs representing points in $\mathbb{D}$ which are moved an admissible distance.

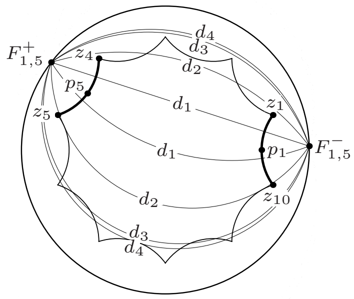


FIGURE 2.7. Some of the admissible arcs for the side-pairing $\gamma_{(1,5)}$ in the particular case $(l, m, n) = (5, 5, 5)$

Consider the surface S underlying the dessin $\mathcal{D}$ of type $(2, 4, 12)$ depicted on the left side of Figure 2.8. The side-pairing in the fundamental polygon makes it clear that the surface has a symmetry of order 2, whose lift to $\mathbb{D}$ is the complex conjugation, and we will only focus on the upper-half.

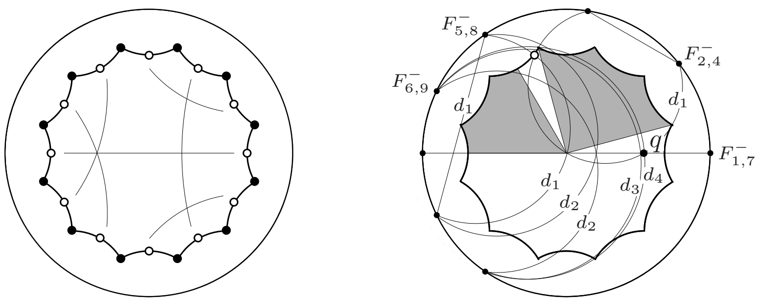


FIGURE 2.8. In S there is a dessin isomorphic to $\mathcal{D}$ centered in $[q]$

It is known (see [31]) that the displacement of the points in a hyperbolic triangle is bounded from above by the displacement of its vertices. Focusing on the upper half of the domain, we find that the identification $\gamma_{(2,4)}$ moves the points z_1, z_4 and 0 exactly the first admissible distance d_1, and the vertices z_2 and z_3 strictly less than d_1. As a consequence there is no admissible point in the interior of the grey region delimited by the vertices z_1, z_2, z_3, z_4 and 0 in Figure 2.8, though perhaps z_1 and z_4 could be admissible points. By the same reason, $\gamma_{(5,8)}$ allows us to get rid of the other grey region, except for z_6. Now, z_1, z_4 and z_6 all correspond to the same point on the surface, but z_4 can be discarded by looking at the set of points moved an admissible distance by $\gamma_{(5,8)}$.

Now we only have to deal with the triangles with vertices at z_4, p_5, 0, and p_1, z_1, 0. In the first one, the only common admissible point for both $\gamma_{(2,4)}$ and $\gamma_{(5,8)}$

is the one depicted in white in Figure 2.8, that is moved a distance between d_2 and d_3 by $\gamma_{(6,9)}$. In the second one there is just one admissible point for $\gamma_{(2,4)}$ and $\gamma_{(1,7)}$, call it q (which is moved a distance d_1 by $\gamma_{(1,7)}$ and $\gamma_{(2,4)}$, and a distance d_4 by $\gamma_{(6,9)}$). We can compute its value $q = \frac{\sqrt{6}}{6}\sqrt[4]{3}$ with hyperbolic methods, relying on the fact that q is translated a distance d_1 by $\gamma_{(2,4)}$.

Since it can be easily checked that the hyperelliptic involution J does not fix the point $[0]$, the point $[q]$ must be the face center of a dessin image under J of $\mathcal{D}$, and therefore isomorphic to it.

Similar arguments regarding admissible distances can be used to study every Belyi surface of genus 2 admitting a unicellular uniform dessin. One gets the following result.

THEOREM 2.5. *Two unicellular uniform dessins of the same type in genus 2 belong to the same surface if and only if they are either dual or isomorphic.*

The details of the proof can be found in [**33**].

CHAPTER 3

Beauville surfaces

"[...] even here, in this region of Three Dimensions, your Lordship's art may make the Fourth Dimension visible to me; just as in the Land of Two Dimensions my Teacher's skill would fain have opened the eyes of his blind servant to the invisible presence of a Third Dimension, though I saw it not."

— Edwin A. Abbott, Flatland: A Romance of Many Dimensions

In this chapter we will be working both with compact Riemann surfaces (manifolds of real dimension two) and complex surfaces (manifolds of real dimension four). As a consequence we will always refer to Riemann surfaces as algebraic curves, and reserve the term surface for complex surfaces.

Beauville surfaces are complex surfaces arising as the quotient of the product of two quasiplatonic curves by an action of a finite group G. To give a more precise definition, let us introduce first the concept of *surfaces isogenous to a product*. These are surfaces X that are isomorphic to the quotient $S_1 \times S_2/G$ of the product of two curves S_1 and S_2 by the free action of a finite group G acting by biholomorphic transformations. If the genus of both curves is $g(S_1), g(S_2) \geq 2$ we say that X is *isogenous to a higher product*.

First of all, let us note that each element of $\mathrm{Aut}(S_1 \times S_2)$ either fixes each curve or interchanges them (see Proposition 0.1). Clearly if two elements $g, h \in G < \mathrm{Aut}(S_1 \times S_2)$ both interchange factors, their product gh does not. In particular if we denote by $G^0 < G$ the subgroup of factor-preserving elements, then $[G : G^0] \leq 2$.

A particular case of surfaces isogenous to a product is Beauville surfaces, introduced by F. Catanese in [14] following a construction of A. Beauville in [11] (see Example 3.1 below).

A *Beauville surface* is a compact complex surface X satisfying the following properties:

(i) X is isogenous to a higher product, $X \cong S_1 \times S_2/G$;
(ii) the subgroup $G^0 < G$ acts effectively on each of the curves S_i producing quotient orbifolds S_i/G^0 of genus zero with three cone points.

We will say that X is of *unmixed type* (or that X is an *unmixed Beauville surface*) if $G = G^0$ and that it is of *mixed type* (or that it is a *mixed Beauville surface*) if $G \neq G^0$. Let us remark that in the mixed case necessarily $S_1 \cong S_2$. If (l_1, m_1, n_1) and (l_2, m_2, n_2) are the types of the G^0-coverings S_1 and S_2, we will say that the Beauville surface $X = S_1 \times S_2/G$ has *bitype* $((l_1, m_1, n_1), (l_2, m_2, n_2))$.

There is always a minimal realization of X in the sense that G^0 acts faithfully on each factor S_i. This is because if, for instance, G^0 did not act faithfully on S_1, so that there exists a subgroup $G' \lhd G$ acting trivially on S_1, then we could write

$$X \cong \frac{S_1 \times S_2}{G} = \frac{S_1/G' \times S_2/G'}{G/G'} = \frac{S_1 \times (S_2/G')}{G/G'} .$$

Hence, from now on we will always assume that the realization $X \cong S_1 \times S_2/G$ is minimal.

Example 3.1 (Beauville). Consider the Fermat curve of degree five

$$F_5 = \{x^5 + y^5 + z^5 = 0\} .$$

The group $G = (\mathbb{Z}/5\mathbb{Z})^2$ acts freely on $F_5 \times F_5$ in the following way: for each $(\alpha, \beta) \in G$ define $e_{\alpha,\beta} : F_5 \times F_5 \longrightarrow F_5 \times F_5$ as

$$\left(\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}, \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} \right) \quad \longmapsto \quad \left(\begin{bmatrix} \zeta^{\alpha} x_1 \\ \zeta^{\beta} y_1 \\ z_1 \end{bmatrix}, \begin{bmatrix} \zeta^{\alpha+3\beta} x_2 \\ \zeta^{2\alpha+4\beta} y_2 \\ z_2 \end{bmatrix} \right) \quad ,$$

where $\zeta = e^{2\pi i/5}$.

Then $X := F_5 \times F_5/(\mathbb{Z}/5\mathbb{Z})^2$ is an unmixed Beauville surface.

Beauville surfaces with abelian Beauville group have been studied and classified ([**14**, **5**, **28**, **40**]). All of them arise as quotients of $F_n \times F_n$ by some action of the group $(\mathbb{Z}/n\mathbb{Z})^2$, where F_n stands for the Fermat curve

$$F_n = \{[x : y : z] \in \mathbb{P}^2(\mathbb{C}) : x^n + y^n + z^n = 0\}$$

and $\gcd(n, 6) = 1$. The number of isomorphism classes of Beauville surfaces which have Beauville group $(\mathbb{Z}/n\mathbb{Z})^2$ is given by a polynomial in n of degree 4 in the case of prime powers, and by a much more complicated formula in the general case (see [**40**]). A consequence of these formulae is that for $n = 5$ there is only one Beauville surface with group $(\mathbb{Z}/5\mathbb{Z})^2$, namely the one above originally constructed by Beauville.

3.1. Uniformisation of unmixed Beauville surfaces

Let now $X = S_1 \times S_2/G$ be a Beauville surface and let us consider first the unmixed case, i.e. the case in which $G = G^0$. Clearly its holomorphic universal cover is the bidisc $\mathbb{H} \times \mathbb{H}$ and the covering group is a subgroup of $\mathrm{Aut}(\mathbb{H} \times \mathbb{H})$. Let us denote it by Γ_{12}, so that $X = \mathbb{H} \times \mathbb{H}/\Gamma_{12}$ with $\Gamma_{12} \cong \pi_1(X)$. The first condition in the definition of Beauville surface implies that there is an exact sequence of the form

$$1 \longrightarrow K_1 \times K_2 \longrightarrow \Gamma_{12} \xrightarrow{\rho} G \longrightarrow 1 \tag{3.1}$$

where K_1 and K_2 uniformise the curves $S_1 = \mathbb{H}/K_1$ and $S_2 = \mathbb{H}/K_2$ and the group $G \cong \Gamma_{12}/K_1 \times K_2$ acts on $S_1 \times S_2$ as follows. Let g be an element of G. If $(\gamma_1, \gamma_2) \in \Gamma_{12}$ is such that $\rho(\gamma_1, \gamma_2) = g$, then the action of g on points $[\mathrm{w}_1, \mathrm{w}_2] \in \mathbb{H}/K_1 \times \mathbb{H}/K_2$ is given by the rule

$$g([\mathrm{w}_1, \mathrm{w}_2]) = [\gamma_1(\mathrm{w}_1), \gamma_2(\mathrm{w}_2)] ,$$

while the action of g on the individual factors is given by $g([\mathrm{w}_1]) = [\gamma_1(\mathrm{w}_1)]$ and $g([\mathrm{w}_2]) = [\gamma_2(\mathrm{w}_2)]$.

Now, by the second condition in the definition, the quotients $\Delta_1 \cong \Gamma_{12}/K_2$ and $\Delta_2 \cong \Gamma_{12}/K_1$ of the group Γ_{12} must be triangle groups defining triangle $G-$covers

$f_i : S_i \cong \mathbb{H}/K_i \longrightarrow \mathbb{P}^1 \cong \mathbb{H}/\Delta_i$ with $G \cong \Delta_i/K_i$. Therefore there are two exact sequences

$$1 \longrightarrow K_i \longrightarrow \Delta_i \xrightarrow{\rho_i} G \longrightarrow 1 \qquad (i = 1, 2)$$

representing the action of G on the individual factors so that, in particular, for the element (γ_1, γ_2) above one must have $\rho_1(\gamma_1) = \rho_2(\gamma_2) = g$. It follows that

$$\Gamma_{12} = \{(\gamma_1, \gamma_2) \in \Delta_1 \times \Delta_2 \ : \ \rho_1(\gamma_1) = \rho_2(\gamma_2)\} < \Delta_1 \times \Delta_2 \,. \tag{3.2}$$

Let (a_i, b_i, c_i) be a generating triple defining the $G-$cover (S_i, f_i). Then the subsets of G

$$\Sigma(a_i, b_i, c_i) := \bigcup_{g\in G} \bigcup_{j=1}^{\infty} \{g a_i^j g^{-1}, g b_i^j g^{-1}, g c_i^j g^{-1}\} \quad (i = 1, 2)$$

consisting of the elements of G that fix points on S_1 and S_2 respectively, necessarily have trivial intersection, that is

$$\Sigma(a_1, b_1, c_1) \cap \Sigma(a_2, b_2, c_2) = \{1\} \,, \tag{3.3}$$

for otherwise the action of G on $S_1 \times S_2$ would not be free.

Conversely, any pair of hyperbolic triples of generators (a_1, b_1, c_1), (a_2, b_2, c_2) of G satisfying condition (3.3) define via the associated epimorphisms ρ_1, ρ_2 a group $\Gamma_{12} < \Delta_1 \times \Delta_2$ as in (3.2), which clearly uniformises a Beauville surface.

Corollary 3.1 ([14]). *Let G be a finite group. Then there are curves S_1 and S_2 of genera $g(S_1), g(S_2) > 1$ and an action of G on $S_1 \times S_2$ so that $S_1 \times S_2/G$ is an unmixed Beauville surface if and only if G has two hyperbolic triples of generators (a_i, b_i, c_i) of order (l_i, m_i, n_i), $i = 1, 2$, satisfying the compatibility condition* (3.3).

Under these assumptions one says that such a pair of triples $(a_1, b_1, c_1; a_2, b_2, c_2)$ is an *unmixed Beauville structure* on G.

This is a useful tool, since it permits to check through a computer program whether or not a group (of not very large order) admits Beauville structure. For instance the following result can be checked by these means

Proposition 3.1. *Let $X = S_1 \times S_2/G$ be a Beauville surface such that the pair of genera $(g(S_1), g(S_2))$ of the curves S_1 and S_2 is at most* $(8, 49)$ *(in the lexicographic order). If G is non-abelian then $G \cong \mathrm{PSL}(2, 7)$.*

Proof. It is known that the minimum possible genus of a curve occurring in the construction of a Beauville surface is 6 ([25]). It is also known that the symmetric group on 5 elements $\mathfrak{S}_5$ is the only non-abelian group up to order 128 admitting a Beauville structure ([5]). The corresponding pair of genera is $(19, 21)$ (see [25]). A list of all groups G acting on a curve S of small genus so that S/G is an orbifold of genus zero with three branching values is given in [17]. There are only six such groups of orders $|G| \geq 128$ acting on curves of genus 6 to 8. A computation carried out with MAGMA for these six groups shows that the only one admitting a Beauville structure is $G = \mathrm{PSL}(2, 7)$ (with pair of genera $(8, 49)$). □

Example 3.2. By the last corollary, corresponding to Beauville's original surface described in Example 3.1 there should be a pair of triples of generators of $G = (\mathbb{Z}/5\mathbb{Z})^2$ of type $(5, 5, 5)$ satisfying the compatibility condition above. In fact the following two triples will do

$$\begin{aligned} a_1 &= (1, 0), \quad b_1 = (0, 1), \quad c_1 = (4, 4), \\ a_2 &= (3, 1), \quad b_2 = (4, 2), \quad c_2 = (3, 2). \end{aligned}$$

The compatibility condition is easily checked, and in fact it is not hard to see that the curve defined by these triples is in both cases the Fermat curve of degree five. To prove this first note that, since all the elements in both triples have order 5, the two corresponding curves will be uniformised by surface subgroups K_1 and K_2 of the triangle group $\Gamma = \Gamma(5,5,5)$. As the quotient $\Gamma/K_i = G$ is abelian, the groups K_i must contain the commutator $[\Gamma,\Gamma]$. But $\Gamma/[\Gamma,\Gamma]$ is already isomorphic to $(\mathbb{Z}/5\mathbb{Z})^2$, so $K_1 = K_2 = [\Gamma,\Gamma]$, and this group is known to uniformise the Fermat curve of degree 5 (see for example [**25, 39**]).

3.1.1. Some restrictions to the existence of unmixed Beauville surfaces. A natural problem regarding Beauville surfaces $X = S_1 \times S_2/G$ is to determine which genera $g(S_1)$ of S_1 and $g(S_2)$ of S_2 can arise in their construction. For instance, in [**25**] it was shown that $g(S_1), g(S_2) \geq 6$. In this section we improve that result.

Perhaps the most direct way to get restrictions on the genera $g(S_1)$ and $g(S_2)$ is to combine Riemann–Hurwitz's formula (1.8) with the formula giving the Euler–Poincaré characteristic of X, namely

$$\chi(X) = \frac{\chi(S_1)\cdot\chi(S_2)}{|G|} = \frac{(2g(S_1)-2)(2g(S_2)-2)}{|G|}, \tag{3.4}$$

the relevant fact being that this fraction has to be a natural number.

Actually an even stronger ingredient is obtained by considering the holomorphic Euler characteristic of X, defined as the alternating sum of the dimensions of the cohomology groups of the structure sheaf, i.e. $\chi(\mathcal{O}_X) = h^0(\mathcal{O}_X) - h^1(\mathcal{O}_X) + h^2(\mathcal{O}_X)$ (see for example [**11**] or [**3**]). In the case of a surface isogenous to a product we have

$$\chi(\mathcal{O}_X) = \frac{(g(S_1)-1)(g(S_2)-1)}{|G|}, \qquad \text{i.e.} \quad \chi(\mathcal{O}_X) = \frac{\chi(X)}{4}\,, \tag{3.5}$$

and the point is, of course, that this fraction is still a natural number.

The last identity follows from Noether's formula, a central result of the theory of complex surfaces, which states that

$$\chi(\mathcal{O}_X) = \frac{1}{12}(K_X^2 + \chi(X))\,.$$

Here, as usual, K_Y^2 denotes the degree of the self-intersection of the canonical class of a complex surface Y. In the particular case in which $Y = S_1 \times S_2$, the degree K_Y^2 can be computed by considering generic holomorphic 1−forms ω_1, ω_1' of S_1 and ω_2, ω_2' of S_2 and looking at the intersection of $\mathcal{Z}(\eta_1)$ and $\mathcal{Z}(\eta_2)$, the zero sets of the 2−forms $\eta_1 = \omega_1 \wedge \omega_2$ and $\eta_2 = \omega_1' \wedge \omega_2'$. Denoting intersection by $\cdot$ and union by $+$, as it is customary in intersection theory, we have

$$\mathcal{Z}(\eta_1)\cdot\mathcal{Z}(\eta_2) = \big(\,(\mathcal{Z}(\omega_1)\times S_2) + (S_1\times\mathcal{Z}(\omega_2))\,\big)\cdot\big(\,(\mathcal{Z}(\omega_1')\times S_2) + (S_1\times\mathcal{Z}(\omega_2'))\,\big),$$

which by the Riemann–Roch theorem for curves is a set consisting of $2(2g(S_1)-2)(2g(S_2)-2)$ points, i.e. $K_Y^2 = 2(2g(S_1)-2)(2g(S_2)-2)$. Therefore for the quotient surface $X = S_1 \times S_2/G$ one has

$$K_X^2 = \frac{2(2g(S_1)-2)(2g(S_2)-2)}{|G|}\,,$$

which gives the expression (3.5) for $\chi(\mathcal{O}_X)$.

Using these ingredients we can prove the following lemma.

LEMMA 3.1. *Let G be an arbitrary finite group and $X = S_1 \times S_2/G$ be an unmixed Beauville surface isogenous to the product of two curves S_1 and S_2 of genera $(g(S_1), g(S_2)) = (p+1, q+1)$ for two prime numbers p and q. Then:*

(i) $p = q$;
(ii) $G = (\mathbb{Z}/n\mathbb{Z})^2$ *for some integer n;*
(iii) $S_1 \cong S_2 \cong F_n$, *the Fermat curve of degree n.*

PROOF. By formula (3.5) the fraction $\chi(\mathcal{O}_X) = pq/|G|$ is a natural number. The only possibility for G being non abelian is to be isomorphic to $\mathbb{Z}/q\mathbb{Z} \rtimes \mathbb{Z}/p\mathbb{Z}$, which can occur only if p divides $q-1$. We claim that in this case G does not admit a Beauville structure.

Indeed, since all $p-$subgroups (resp. $q-$subgroups) of $\mathbb{Z}/q\mathbb{Z} \rtimes \mathbb{Z}/p\mathbb{Z}$ are conjugate, then any possible pair of generating triples (a_1, b_1, c_1) and (a_2, b_2, c_2) satisfying the compatibility condition (3.3) must have orders (p, p, p) and (q, q, q) respectively. Now the image $\overline{x} \in G/(\mathbb{Z}/q\mathbb{Z})$ of any element $x \in G$ of order q can only be the identity, and so $x \in \mathbb{Z}/q\mathbb{Z}$. In other words, no triple of elements of order q such as (a_2, b_2, c_2) can generate the whole group G. Therefore G must be abelian, and by [14] necessarily $p = q$ and $G = (\mathbb{Z}/p\mathbb{Z})^2$.

Now, arguing as in Example 3.2, we can deduce that both curves S_1 and S_2 are isomorphic to the Fermat curve of degree p. □

In fact there are no Fermat curves of genus $p+1$ for any prime $p > 5$. This is only because the genus of F_d is $g = (d-1)(d-2)/2$, which cannot equal $p+1$ for any prime $p > 5$.

THEOREM 3.1. *If $X = S_1 \times S_2/G$ is an unmixed Beauville surface with pair of genera $(g(S_1), g(S_2)) = (p+1, q+1)$, for prime numbers p and q, then $p = q = 5$ and X is isomorphic to the complex surface described in Example 3.1. In particular, this is the only Beauville surface reaching the minimum possible pair of genera $(6, 6)$.*

The next pair of genera (in the lexicographic order) for which there exists a Beauville surface is $(8, 49)$, therefore there are no Beauville surfaces with pair of genera $(6, g(S_2))$ or $(7, g(S_2))$ for any $g(S_2) > 6$.

PROOF. The first part of the theorem follows directly from the previous comments and the already mentioned fact that Beauville's original example described in Example 3.1 is the only Beauville surface with group $(\mathbb{Z}/5\mathbb{Z})^2$. The second one is consequence of Proposition 3.1. □

3.1.2. Isomorphisms of unmixed Beauville surfaces. Let us suppose that there is an isomorphism f between two Beauville surfaces X and X'. By covering space theory we can lift f to an isomorphism between their universal coverings to obtain a commutative diagram as follows

$$\begin{array}{ccccc} & \mathbb{H}\times\mathbb{H} & \xrightarrow{\tilde{f}} & \mathbb{H}\times\mathbb{H} & \\ & \downarrow & & \downarrow & \\ X \;= & \dfrac{\mathbb{H}\times\mathbb{H}}{\Gamma_{12}} & \xrightarrow{f} & \dfrac{\mathbb{H}\times\mathbb{H}}{\Gamma'_{12}} & =\; X' \end{array}$$

By Proposition 0.1, there exist $\tilde{f}_1, \tilde{f}_2 \in \mathrm{PSL}(2, \mathbb{R})$ such that

$$\tilde{f}(\mathrm{w}_1, \mathrm{w}_2) = \begin{cases} (\tilde{f}_1(\mathrm{w}_1), \tilde{f}_2(\mathrm{w}_2)), & \text{if } \tilde{f} \text{ does not interchange factors,} \\ (\tilde{f}_1(\mathrm{w}_2), \tilde{f}_2(\mathrm{w}_1)), & \text{if } \tilde{f} \text{ interchanges factors.} \end{cases}$$

Note that in the second case $\tilde{f}$ can be rewritten as $\tilde{f} = (\tilde{f}_1, \tilde{f}_2) \circ J$.

Bauer, Catanese and Grunewald proved in [5] the following characterisation of isomorphism classes of Beauville surfaces.

Proposition 3.2. *Two unmixed Beauville surfaces X and X' are isomorphic if and only if there exist $\delta_1, \delta_2 \in \mathrm{PSL}(2,\mathbb{R})$, $\psi \in \mathrm{Aut}(G)$ and a permutation $\nu \in \mathfrak{S}_2$ such that the following diagrams commute*

$$\begin{array}{ccc} \Delta_1 & \xrightarrow{\varphi_{\delta_1}} & \Gamma'_{\nu(1)} \\ \rho_1 \downarrow & & \downarrow \rho'_{\nu(1)} \\ G & \xrightarrow{\psi} & G \end{array} \qquad \begin{array}{ccc} \Delta_2 & \xrightarrow{\varphi_{\delta_2}} & \Gamma'_{\nu(2)} \\ \rho_2 \downarrow & & \downarrow \rho'_{\nu(2)} \\ G & \xrightarrow{\psi} & G \end{array} \tag{3.6}$$

i.e. such that $\psi \circ \rho_i = \rho'_{\nu(i)} \circ \varphi_{\delta_i}$.

The proof is based in the fact that the lift of such an isomorphism must conjugate the subgroups $K_1, K_2 < \Gamma_{12}$ into the subgroups $K'_1, K'_2 < \Gamma'_{12}$.

We can translate this proposition into conditions on the pairs of triples of generators of G for their corresponding Beauville surfaces to be isomorphic.

Corollary 3.2. *Let $q = (a_1, b_1, c_1; a_2, b_2, c_2)$ and $q' = (a'_1, b'_1, c'_1; a'_2, b'_2, c'_2)$ be two Beauville structures on G. Then the Beauville surfaces corresponding to q and q' are isomorphic if and only if there exists $\psi \in \mathrm{Aut}(G)$ and $\nu \in \mathfrak{S}_2$ such that*

$$\psi(a_i, b_i, c_i) \equiv (a'_{\nu(i)}, b'_{\nu(i)}, c'_{\nu(i)}) \bmod \mathrm{I}(G; l'_{\nu(i)}, m'_{\nu(i)}, n'_{\nu(i)})\,, \quad i = 1, 2. \tag{3.7}$$

Moreover, the corresponding uniformising groups are conjugate by means of any element $(\delta_1, \delta_2) \in \mathrm{Aut}(\mathbb{H}) \times \mathrm{Aut}(\mathbb{H})$ fitting into (3.6).

By the comments above we have the following

Corollary 3.3. *The following are invariants of the isomorphism class of an unmixed Beauville surface $X = S_1 \times S_2/G$:*

(i) *the group G;*
(ii) *the bitype $((l_1, m_1, n_1), (l_2, m_2, n_2))$;*
(iii) *the twisted isomorphism class of the orbifolds S_i/G, hence the curves S_i themselves.*

3.1.3. Automorphisms of unmixed Beauville surfaces. In this section we will study the group of automorphisms of unmixed Beauville surfaces. If we denote by $\Gamma_{12} < \mathrm{Aut}(\mathbb{H}) \times \mathrm{Aut}(\mathbb{H})$ the group uniformising such a Beauville surface X, as described in (3.1) and (3.2), then of course $\mathrm{Aut}(X) \cong N(\Gamma_{12})/\Gamma_{12}$, where $N(\Gamma_{12})$ stands for the normaliser of Γ_{12} in $\mathrm{Aut}(\mathbb{H} \times \mathbb{H})$.

Consider first the subgroup $N(\Gamma_{12}) \cap (\Delta_1 \times \Delta_2)$. We have the following result.

Lemma 3.2. *The rule*

$$\begin{array}{rccc} \theta: & N(\Gamma_{12}) \cap (\Delta_1 \times \Delta_2) & \longrightarrow & Z(G) \\ & (\gamma_1, \gamma_2) & \longmapsto & \rho_2(\gamma_2)^{-1}\rho_1(\gamma_1) \end{array}$$

defines an epimorphism whose kernel is Γ_{12}. Here, as usual, $Z(G)$ stands for the centre of G.

Proof. We first observe that an element $(\gamma_1, \gamma_2) \in \Delta_1 \times \Delta_2$ normalises Γ_{12} if and only if for every $g \in G$ one has

$$\rho_1(\gamma_1) g \rho_1(\gamma_1)^{-1} = \rho_2(\gamma_2) g \rho_2(\gamma_2)^{-1}\,, \tag{3.8}$$

i.e. $\rho_2(\gamma_2)^{-1}\rho_1(\gamma_1) \in Z(G)$. This shows that the map θ is well defined.

Now it is easy to see that θ is a homomorphism. Indeed

$$\begin{aligned}\theta\left((\gamma_1,\gamma_2)\circ(\gamma_1',\gamma_2')\right) &= \theta\left(\gamma_1\gamma_1',\gamma_2\gamma_2'\right) = \rho_2(\gamma_2')^{-1}\rho_2(\gamma_2)^{-1}\rho_1(\gamma_1)\rho_1(\gamma_1') = \\ &= \rho_2(\gamma_2')^{-1}\theta\left(\gamma_1,\gamma_2\right)\rho_1(\gamma_1') = \theta\left(\gamma_1,\gamma_2\right)\cdot\theta\left(\gamma_1',\gamma_2'\right) .\end{aligned}$$

On the other hand, if $\rho_1(\beta) = h \in Z(G)$ then the element $(\beta, 1)$ clearly satisfies the relation (3.8) and therefore it is a preimage of h.

Finally, we see that $\theta(\gamma_1, \gamma_2) = 1$ if and only if $\rho_1(\gamma_1) = \rho_2(\gamma_2)$, that is if and only if $(\gamma_1, \gamma_2) \in \Gamma_{12}$. □

Now we can prove the following

THEOREM 3.2. *Let X be an unmixed Beauville surface with Beauville group G. The group $Z(G)$ is naturally identified with a subgroup of* Aut(X) *of index dividing 72. More precisely, let X have bitype $((l_1, m_1, n_1), (l_2, m_2, n_2))$, and consider natural numbers ε, κ_1, and κ_2 where ε equals 2 if the types (l_1, m_1, n_1) and (l_2, m_2, n_2) agree and 1 otherwise, and κ_i equals 6, 2 or 1 depending on whether the type (l_i, m_i, n_i) has three, two or no repeated orders. Then there exists a natural number N dividing $\varepsilon \cdot \kappa_1 \cdot \kappa_2$ such that*

$$|\mathrm{Aut}(X)| = N \cdot |Z(G)| .$$

In particular, if $\kappa_1 = \kappa_2 = \varepsilon = 1$ we have that $\mathrm{Aut}(X) \cong Z(G)$.

PROOF. The previous lemma permits us to regard $Z(G)$ as a subgroup of Aut(X) via the identification

$$Z(G) \cong \frac{N(\Gamma_{12}) \cap (\Delta_1 \times \Delta_2)}{\Gamma_{12}} \leq \mathrm{Aut}(X) .$$

Consider the intersections

$$\begin{aligned} N_0(\Gamma_{12}) &= N(\Gamma_{12}) \cap (\mathrm{Aut}(\mathbb{H}) \times \mathrm{Aut}(\mathbb{H})) \qquad \text{and} \\ N_1(\Gamma_{12}) &= N_0(\Gamma_{12}) \cap (\Delta_1 \times \Delta_2) = N(\Gamma_{12}) \cap (\Delta_1 \times \Delta_2) . \end{aligned}$$

Using the identity $|\mathrm{Aut}(X)| = [N(\Gamma_{12}) : \Gamma_{12}]$ one gets the following equality

$$|\mathrm{Aut}(X)| = [N(\Gamma_{12}) : N_0(\Gamma_{12})] \cdot [N_0(\Gamma_{12}) : N_1(\Gamma_{12})] \cdot [N_1(\Gamma_{12}) : \Gamma_{12}] .$$

Now, $\varepsilon := [N(\Gamma_{12}) : N_0(\Gamma_{12})] \leq 2$ and $[N_1(\Gamma_{12}) : \Gamma_{12}] = |Z(G)|$.

On the other hand, clearly one has $N_0(\Gamma_{12}) < N(\Delta_1) \times N(\Delta_2)$, and therefore $[N_0(\Gamma_{12}) : N_1(\Gamma_{12})]$ divides $[N(\Delta_1) \times N(\Delta_2) : \Delta_1 \times \Delta_2]$. If we write $\kappa_i := |N(\Delta_i)/\Delta_i|$, then $[N(\Delta_1) \times N(\Delta_2) : \Delta_1 \times \Delta_2] = \kappa_1 \cdot \kappa_2$ and the result follows from (1.4). □

The above bounds are actually sharp, as shown by examples by Y. Fuertes ([23]) and by G. A. Jones in [47]. This last paper contains most of what is known about the automorphism groups of unmixed Beauville surfaces.

EXAMPLE 3.3. For Beauville's original surface with group $G = (\mathbb{Z}/5\mathbb{Z})^2$ and bitype $((5, 5, 5), (5, 5, 5))$, the automorphism group is a semidirect product of the centre $Z(G) = G$ and $\mathbb{Z}/3\mathbb{Z}$ ([40]), and therefore $|\mathrm{Aut}(X)| = 3 \cdot |Z(G)| = 75$.

3.2. Uniformisation of mixed Beauville surfaces

We focus our attention now on the mixed case. Recall that a mixed Beauville surface is a surface of the form $X = S_1 \times S_2/G$, where G is a finite group acting freely on $S_1 \times S_2$ so that the index two subgroup $G^0 \triangleleft G$ of factor-preserving elements of G acts on each of the two Riemann surfaces in such a way that the projections $S_i \longrightarrow S_i/G^0 \cong \widehat{\mathbb{C}}$ ramify over three values. Note that if $g \in G \backslash G^0$ then $G = \langle G^0, g\rangle$ and, moreover, the action of g defines a factor-reversing automorphism of the associated unmixed Beauville surface $X^0 = S_1 \times S_2/G^0$. Such an element g induces an isomorphism between the orbifolds S_1/G^0 and S_2/G^0. It follows that in this case $S_1 \cong S_2$, and that the corresponding triangle groups Δ_1 and Δ_2 are both equal to the group $\Delta = \Delta(l, m, n)$. As a consequence in the mixed case instead of the bitype we will simply call (l, m, n) the *type of* X.

Uniformisation theory tells us that there is a group $\Gamma_{12} < \mathrm{Aut}(\mathbb{H} \times \mathbb{H})$ such that $X = \mathbb{H} \times \mathbb{H}/\Gamma_{12}$ and $X^0 = \mathbb{H} \times \mathbb{H}/\Gamma_{12}^0$ where $\Gamma_{12}^0 < \Delta \times \Delta$ is the index two subgroup of Γ_{12} consisting of the factor-preserving elements. Therefore we have exact sequences

$$\begin{array}{ccccccccc} 1 & \longrightarrow & K_1 \times K_2 & \longrightarrow & \Gamma_{12} & \xrightarrow{\rho} & G & \longrightarrow & 1 \\ 1 & \longrightarrow & K_1 \times K_2 & \longrightarrow & \Gamma_{12}^0 & \xrightarrow{\rho^0} & G^0 & \longrightarrow & 1 \end{array}$$

where $\rho^0((\gamma_1, \gamma_2)) = \rho_1(\gamma_1) = \rho_2(\gamma_2)$ and $\rho|_{\Gamma_{12}^0} = \rho^0$. In particular the epimorphism ρ is determined by ρ^0 together with the image $\rho(\mathfrak{h}) = h \in G$ of any chosen element $\mathfrak{h} \in \Gamma_{12} \setminus \Gamma_{12}^0$.

Note that each element $\mathfrak{h} \in \Gamma_{12} \setminus \Gamma_{12}^0$ can be written as $\mathfrak{h} = (\beta_1, \beta_2) \circ J$ where $\beta_1, \beta_2 \in \mathrm{Aut}(\mathbb{H})$. Now, as $\mathfrak{h}$ must normalise Γ_{12}^0, for every element $(\gamma_1, \gamma_2) \in \Gamma_{12}^0$ we have

$$\begin{aligned} \mathfrak{h} \circ (\gamma_1, \gamma_2) \circ \mathfrak{h}^{-1} &= (\beta_1, \beta_2) \circ J \circ (\gamma_1, \gamma_2) \circ J \circ (\beta_1^{-1}, \beta_2^{-1}) = \\ &= (\beta_1 \gamma_2 \beta_1^{-1}, \beta_2 \gamma_1 \beta_2^{-1}) \in \Gamma_{12}^0 \,. \end{aligned}$$

It follows that $\beta_1, \beta_2 \in N(\Delta)$, the normaliser of $\Delta = \Delta(l, m, n)$.

With these facts one can get a criterion for mixed surfaces analogous to the one established in Corollary 3.1 for the unmixed ones. As in the unmixed case we will say that a finite group G admits a *mixed Beauville structure* if there exists an action of G on the product of two Riemann surfaces defining a mixed Beauville surface.

PROPOSITION 3.3. *A finite group G admits a mixed Beauville structure if and only if there exist an index two subgroup $G^0 \triangleleft G$ and elements $a, b, c \in G^0$ such that the following conditions hold:*

(i) *(a, b, c) is a hyperbolic triple of generators of G^0;*
(ii) *$h^2 \neq \mathrm{Id}$, for every $h \in G \setminus G^0$;*
(iii) *there exists $g \in G \backslash G^0$ such that $\Sigma(a, b, c) \cap \Sigma(gag^{-1}, gbg^{-1}, gcg^{-1}) = \{\mathrm{Id}\}$.*

REMARK 3.1. It is important to observe that if, in the construction above, instead of the element g we use another element $g' \in G \setminus G^0$ satisfying condition (iii) in Proposition 3.3, then the mixed Beauville surface X' so obtained will be isomorphic to X.

Due to the remark above we can refer to a mixed Beauville structure on G simply by giving a quadruple $(G^0; a, b, c)$ satisfying the conditions in Proposition 3.3, without need to mention any particular element $g \in G \setminus G^0$.

3.2.1. Some restrictions to the existence of mixed Beauville surfaces. There are some obvious conditions that groups admitting mixed Beauville structures must satisfy. For instance, simple groups cannot do so, as they do not possess index two subgroups. Likewise, the symmetric groups $\mathfrak{S}_n$ do not admit mixed Beauville structures either. This is because the only subgroup of $\mathfrak{S}_n$ of index two is the alternating group $\mathfrak{A}_n$, and $\mathfrak{S}_n \setminus \mathfrak{A}_n$ contains plenty of elements of order two, a fact which violates condition (ii) in Proposition 3.3. Another family of groups which cannot admit mixed Beauville structures is the abelian ones (see [5], Theorem 4.3).

The next result included in [24] exhibits another restriction of this sort.

PROPOSITION 3.4. *Let G be a group admitting a Beauville structure. Then the order of any element of $G \setminus G^0$ is divisible by 4. In particular, the order $|G|$ of G is a multiple of 4.*

PROOF. Let $g \in G \setminus G^0$ an element of order k. If k is an odd natural number then g^k is still factor-reversing, thus different from the identity. Therefore k is necessarily even. Now if $k = 2d$, then $(g^d)^2 = 1$ which by condition (ii) in Proposition 3.3 implies that $g^d \in G^0$, which in turn implies that d is even. □

Next we give a restriction on the genus of the Riemann surfaces that can arise in the construction of mixed Beauville surfaces.

Since both Riemann surfaces S_1, S_2 intervening in the construction of a mixed Beauville surface are isomorphic to the same Riemann surface $S \cong S_1 \cong S_2$, using the formulae (3.4) and (3.5) for the Euler–Poincaré characteristic and the holomorphic Euler characteristic we get

$$\chi(\mathcal{O}_X) = \frac{\chi(X)}{4} = \frac{(g(S)-1)^2}{|G|} = \frac{(g(S)-1)^2}{2|G^0|} \in \mathbb{N},$$

where $g(S)$ is the genus of the Riemann surface S. Thus, in particular, $g(S)$ is odd. This formula already tells us that $(g(S)-1)^2 \geq |G|$.

On the other hand, by the Riemann–Hurwitz formula we have

$$2g(S) - 2 = |G^0| \left(1 - \left(\frac{1}{l} + \frac{1}{m} + \frac{1}{n}\right)\right),$$

where (l, m, n) is the signature of the $G-$covering S. Furthermore, it is known that $1/42 \leq 1 - (1/l + 1/m + 1/n) < 1$ and therefore, from the last two formulae we can deduce that

$$\max\left\{\sqrt{|G|} + 1, \frac{|G|}{168} + 1\right\} \leq g(S) < \frac{|G|}{4} + 1. \tag{3.9}$$

Now it is known that no group of order smaller than 256 admits a mixed Beauville structure. In fact, in [7] it is proved that there are two groups of order 256 admitting a mixed Beauville structure of type $(4, 4, 4)$, whose corresponding Riemann surfaces have genus 17. This fact together with the lower bound in (3.9) leads to the following.

COROLLARY 3.4. *Let $X = S \times S/G$ be a mixed Beauville surface. Then $g(S)$ is an odd number ≥ 17 and this bound is sharp.*

Proof. We already noted that $g(S)$ has to be odd.

Moreover, the comments above together with the inequalities in (3.9) imply that $g(S) \geq \max\left\{\sqrt{256}+1, \frac{256}{168}+1\right\} = 17$. $\square$

3.2.2. Isomorphisms of mixed Beauville surfaces. Let us consider two mixed Beauville surfaces $X = S \times S/G$ and $X' = S' \times S'/G'$, associated to mixed Beauville structures $(G^0; a, b, c)$ and $(G'^0; a', b', c')$, and having underlying unmixed Beauville surfaces X^0 and X'^0, respectively.

Suppose $f : X \longrightarrow X'$ is an isomorphism. Let $\tilde{f} : \mathbb{H} \times \mathbb{H} \longrightarrow \mathbb{H} \times \mathbb{H}$ be its lift to the universal cover and $f_* : \Gamma_{12} \longrightarrow \Gamma'_{12}$ the group isomorphism induced by $\tilde{f}$. Clearly the restriction $f_*|_{\Gamma^0_{12}}$ gives an isomorphism between Γ^0_{12} and Γ'^0_{12}. In particular f lifts to an isomorphism $f^0 : X^0 \longrightarrow X'^0$ and we have the following commutative diagram

$$\begin{array}{ccccccc}
 & & \mathbb{H} \times \mathbb{H} & \xrightarrow{\ \tilde{f}\ } & \mathbb{H} \times \mathbb{H} & & \\
 & & \downarrow & & \downarrow & & \\
X^0 & = & \dfrac{\mathbb{H} \times \mathbb{H}}{\Gamma^0_{12}} & \xrightarrow{\ f^0\ } & \dfrac{\mathbb{H} \times \mathbb{H}}{\Gamma'^0_{12}} & = & X'^0 \\
 & & \downarrow & & \downarrow & & \\
X & = & \dfrac{\mathbb{H} \times \mathbb{H}}{\Gamma_{12}} & \xrightarrow{\ f\ } & \dfrac{\mathbb{H} \times \mathbb{H}}{\Gamma'_{12}} & = & X'
\end{array}$$

Moreover, as in the unmixed case $\tilde{f}$ conjugates $K_1 \times K_2$ to $K'_1 \times K'_2$, therefore it induces an isomorphism ψ between $G \cong \Gamma_{12}/K_1 \times K_2$ and $G' \cong \Gamma'_{12}/K'_1 \times K'_2$ which restricts to an isomorphism between $G^0 \cong \Gamma^0_{12}/K_1 \times K_2$ and $G'^0 \cong \Gamma'^0_{12}/K'_1 \times K'_2$.

By pre-composition with an element of Γ_{12} if necessary, we can always assume that $\tilde{f}$ is factor-preserving. Then, with the same notation as in section 3.1.2 (except that here G^0 plays the role of the group we denoted there by G), one has (see (3.7))

$$\psi(a, b, c) \equiv (a', b', c') \bmod \mathrm{I}(G'^0; l, m, n)\,.$$

Conversely, it can be seen that the existence of an isomorphism $\psi : G \longrightarrow G'$, with $\psi(G^0) = G'^0$, such that $\psi(a, b, c) \equiv (a', b', c') \bmod \mathrm{I}(G'^0; l, m, n)$ implies that the groups Γ_{12} and Γ'_{12} uniformising the mixed Beauville surfaces corresponding to the quadruples $(G^0; a, b, c)$ and $(G'^0; a', b', c')$ are conjugate.

Therefore we have the following characterization of isomorphism classes of mixed Beauville surfaces via their defining quadruples.

Corollary 3.5. *Let $q = (G^0; a, b, c)$ and $q' = (G'^0; a', b', c')$ be Beauville structures on G. Then the Beauville surfaces corresponding to q and q' are isomorphic if and only if there exists an automorphism ψ of G with $\psi(G^0) = G'^0$ such that*

$$\psi(a, b, c) \equiv (a', b', c') \bmod \mathrm{I}(G'^0; l, m, n)\,.$$

Corollary 3.6. *The following are invariants of the isomorphism class of a mixed Beauville surface $X = S \times S/G$:*

(i) *the abstract groups G and G^0;*
(ii) *the type (l, m, n) of X;*
(iii) *the twisted isomorphism class of the G^0-covering $S \longrightarrow S/G^0$, hence the Riemann surface S itself.*

3.2.3. Automorphisms of mixed Beauville surfaces. Proceeding in the same way as in section 3.1.3, we will study the group of automorphisms of a mixed Beauville surface X. We have the following chain of inclusions

$$\Gamma_{12}^0 \lhd \Gamma_{12} < N(\Gamma_{12}) < N(\Gamma_{12}^0) < \mathrm{Aut}(\mathbb{H} \times \mathbb{H}),$$

and the automorphism group of X can be seen as $\mathrm{Aut}(X) \cong N(\Gamma_{12})/\Gamma_{12}$. Consider the intersections

$$\begin{aligned} N_0(\Gamma_{12}) &= N(\Gamma_{12}) \cap (\mathrm{Aut}(\mathbb{H}) \times \mathrm{Aut}(\mathbb{H})) \qquad \text{and} \\ N_1(\Gamma_{12}) &= N_0(\Gamma_{12}) \cap (\Delta \times \Delta) = N(\Gamma_{12}) \cap (\Delta \times \Delta). \end{aligned}$$

We have a natural isomorphism

$$N_0(\Gamma_{12})/\Gamma_{12}^0 \cong N(\Gamma_{12})/\Gamma_{12} \tag{3.10}$$

induced by the natural injection of $N_0(\Gamma_{12})$ in $N(\Gamma_{12})$.

As in the unmixed case (section 3.1.3) we have a homomorphism

$$\begin{aligned} \theta: \quad N_1(\Gamma_{12}) &\longrightarrow Z(G^0) \\ (\gamma_1, \gamma_2) &\longmapsto \rho_2(\gamma_2)^{-1}\rho_1(\gamma_1) \end{aligned}$$

whose kernel is Γ_{12}^0.

Choose an element $g \in G \setminus G^0$ and define the subgroup

$$Z(G^0)_{-1} := \{h \in Z(G^0) : gh^{-1}g^{-1} = h\}.$$

As any other element of $G \setminus G^0$ is of the form $g' = gh_0$ for some $h_0 \in G^0$, one readily sees that $Z(G^0)_{-1}$ does not depend on the choice of g within the subset $G \setminus G^0$. We claim that $\mathrm{Im}(\theta) = Z(G^0)_{-1}$.

Now recall that a uniformising group of X was provided by $\Gamma_{12} = \langle \Gamma_{12}^0, \mathfrak{g}\rangle$, where $\mathfrak{g} = (\tau, 1) \circ J$ for any $\tau \in \Delta$ with $\rho_1(\tau) = g^2$. Therefore any element $(\gamma_1, \gamma_2) \in N(\Gamma_{12})$ must satisfy

$$(\gamma_1, \gamma_2) \circ (\tau, 1) \circ J \circ (\gamma_1, \gamma_2)^{-1} = (\gamma_1\tau\gamma_2^{-1}\tau^{-1}, \gamma_2\gamma_1^{-1}) \circ (\tau, 1) \circ J \in \Gamma_{12},$$

i.e. $(\gamma_1\tau\gamma_2^{-1}\tau^{-1}, \gamma_2\gamma_1^{-1}) \in \Gamma_{12}^0$, which is equivalent to the equality

$$\rho_2(\gamma_2)\rho_2(\gamma_1)^{-1} = \rho_1(\gamma_1\tau\gamma_2^{-1}\tau^{-1}) = \rho_1(\gamma_1)g^2\rho_1(\gamma_2)^{-1}g^{-2}. \tag{3.11}$$

Now pre-multiplying both sides by $\rho_2(\gamma_2)^{-1}$ and bearing in mind that the element $\rho_2(\gamma_2)^{-1}\rho_1(\gamma_1)$ belongs to $Z(G^0)$ we get

$$\rho_2(\gamma_1)^{-1} = g^2\rho_2(\gamma_2)^{-1}\rho_1(\gamma_1)\rho_1(\gamma_2)^{-1}g^{-2}$$

and then, using the relation $\rho_2(\gamma_1)^{-1} = g\rho_1(\gamma_1)^{-1}g^{-1}$ and taking inverses we obtain

$$\rho_1(\gamma_1) = g\rho_1(\gamma_2)\rho_1(\gamma_1)^{-1}\rho_2(\gamma_2)g^{-1} = g\rho_1(\gamma_2)g^{-1} \cdot g\theta(\gamma_1, \gamma_2)^{-1}g^{-1}.$$

Finally, since $g\rho_1(\gamma_2)g^{-1} = \rho_2(\gamma_2)$, from these equalities one easily gets

$$\theta(\gamma_1, \gamma_2) = g \cdot \theta(\gamma_1, \gamma_2)^{-1} \cdot g^{-1},$$

hence $\theta(\gamma_1, \gamma_2) \in Z(G^0)_{-1}$.

To prove that θ is an epimorphism take any $h \in Z(G^0)_{-1}$ and let $\gamma \in \Delta$ be such that $\rho_1(\gamma) = h$. Then $\rho_2(\gamma^{-1}) = g\rho_1(\gamma^{-1})g^{-1} = gh^{-1}g^{-1} = h$, hence $(\gamma, \gamma^{-1}) \in \Gamma_{12}^0$ which in turn implies that $(\gamma, 1) \in N_1(\Gamma_{12})$ since it satisfies formula (3.11), and clearly $\theta(\gamma, 1) = h$. Therefore we have

$$\frac{N_1(\Gamma_{12})}{\Gamma_{12}^0} \cong Z(G^0)_{-1}$$

which can be regarded as a subgroup of $\mathrm{Aut}(X) = N(\Gamma_{12})/\Gamma_{12}$ via the identification (3.10).

Now we can prove the following:

THEOREM 3.3. *Let X be a mixed Beauville surface with group G. The group $Z(G^0)_{-1}$ is canonically identified with a subgroup of $\mathrm{Aut}(X)$ of index dividing 36. More precisely, let κ be 6, 2 or 1 depending on whether the type (l, m, n) of X has three, two or no repeated orders. Then there exists a natural number N dividing κ^2 such that*

$$|\mathrm{Aut}(X)| = N \cdot |Z(G^0)_{-1}|\,.$$

In particular, if $\kappa = 1$ then $\mathrm{Aut}(X) \cong Z(G^0)_{-1}$.

PROOF. By (3.10) one has the following equality

$$|\mathrm{Aut}(X)| = |N_0(\Gamma_{12})/\Gamma_{12}^0| = [N_0(\Gamma_{12}) : N_1(\Gamma_{12})] \cdot [N_1(\Gamma_{12}) : \Gamma_{12}^0]\,.$$

Now, by the comments above we have $[N_1(\Gamma_{12}) : \Gamma_{12}^0] = |Z(G^0)_{-1}|$.
On the other hand $N_0(\Gamma_{12}) < N(\Delta) \times N(\Delta)$, and so $[N_0(\Gamma_{12}) : N_1(\Gamma_{12})]$ divides $[N(\Delta) \times N(\Delta) : \Delta \times \Delta] = |N(\Delta)/\Delta|^2 = \kappa^2$, and the result follows from (1.4). □

3.3. Metric rigidity of unmixed Beauville surfaces

In this section we prove an alternative version of Catanese's rigidity results for Beauville surfaces. We recall that, if we consider the metric in $\mathbb{H} \times \mathbb{H}$ given by the Pythagorean formula $ds^2_{\mathbb{H}\times\mathbb{H}} = (ds_1)^2_{\mathbb{H}} + (ds_2)^2_{\mathbb{H}}$, the automorphisms of $\mathbb{H} \times \mathbb{H}$ coincide with its isometries, and therefore the group of factor-preserving isometries of $\mathbb{H} \times \mathbb{H}$ agrees with $\mathrm{Aut}(\mathbb{H}) \times \mathrm{Aut}(\mathbb{H})$, which contains the uniformising group Γ_{12}. Therefore any Beauville surface carries a canonical metric induced by the product metric on $\mathbb{H} \times \mathbb{H}$.

The following rigidity theorem for Beauville surfaces is a consequence of the rigidity of triangle groups.

THEOREM 3.4. *Two unmixed Beauville surfaces X and X' are isometric if and only if $\pi_1(X) \cong \pi_1(X')$.*

PROOF. Let us identify the fundamental groups of X and X' with their corresponding uniformising groups $\Gamma_{12}, \Gamma'_{12} < \mathrm{Aut}(\mathbb{H}) \times \mathrm{Aut}(\mathbb{H})$ and let $\Phi : \Gamma_{12} \longrightarrow \Gamma'_{12}$ be a group isomorphism. First we claim that, up to renumbering, $\Phi(K_1) = K'_1$ and $\Phi(K_2) = K'_2$ so that, in particular, $\Phi(K_1 \times K_2) = K'_1 \times K'_2$. Clearly the centraliser $\mathcal{C}_{\Gamma_{12}}((\gamma_1, \gamma_2))$ of an element $(\gamma_1, \gamma_2) \in \Gamma_{12}$ agrees with $(\mathcal{C}_{\Delta_1}(\gamma_1) \times \mathcal{C}_{\Delta_2}(\gamma_2)) \cap \Gamma_{12}$, and it is known that $\mathcal{C}_{\Delta_i}(\gamma_i)$ is abelian if $\gamma_i \neq 1$ (see for instance [**32**], Remark 2.3). Therefore $\mathcal{C}_{\Gamma_{12}}((\gamma_1, \gamma_2))$ is either abelian, when $\gamma_i \neq 1$ for $i = 1, 2$, or contains the subgroup K_i, hence is not abelian, if $\gamma_i = 1$. As a consequence one of the coordinates of the image of any element $(k, 1) \in K_1$ must be 1, say the second one, and therefore $\Phi(K_1) = K'_1$.

Moreover, since clearly $\Delta_1 \cong \Gamma_{12}/K_2$ and $\Delta_2 \cong \Gamma_{12}/K_1$, it further follows that Φ induces isomorphisms $\Phi_i : \Delta_i \longrightarrow \Delta'_i$ defined by

$$\Phi_1(\gamma_1) = p_1 \circ \Phi(\gamma_1, \gamma_2)\,,$$

where p_1 stands for the first projection and γ_2 is any element of Δ_2 so that $(\gamma_1, \gamma_2) \in \Gamma_{12}$. In other words the isomorphism $\Phi : \Gamma_{12} \longrightarrow \Gamma'_{12}$ extends to an isomorphism $\Phi_1 \times \Phi_2 : \Delta_1 \times \Delta_2 \longrightarrow \Delta'_1 \times \Delta'_2$.

Now by Corollary 1.1 any group isomorphism between triangle groups is induced by an isometry of $\mathbb{H}$ and therefore the product of the isometries δ_1, δ_2 corresponding to Φ_1, Φ_2 induces the required isometry $\delta_1 \times \delta_2 : X \longrightarrow X'$. □

As a corollary we obtain

THEOREM 3.5 (Catanese [14, 6]). *Let $X = S_1 \times S_2/G$ be a Beauville surface.*

(i) *If $X' = S_1' \times S_2'/G'$ is another Beauville surface such that $\pi_1(X) = \pi_1(X')$ then, up to renumbering, $S_i' \cong S_i$ or $\overline{S_i}$ for $i = 1, 2$.*

(ii) *There are at most four non-isomorphic Beauville surfaces with fundamental group isomorphic to $\pi_1(X)$.*

PROOF. *(i)* The isomorphisms between K_i and K_i' in the previous proof are induced by isometries δ_i. Thus, depending on whether these are orientation-preserving or orientation-reversing, we have $S_i' \cong S_i$ or $S_i' \cong \overline{S_i}$.

(ii) Let $\delta_1 \times \delta_2 : X \longrightarrow X'$ be an isometry between X and any other Beauville surface X' with same fundamental group. If both isometries δ_i are simultaneously orientation-preserving then $\delta_1 \times \delta_2 : X \longrightarrow X'$ is a holomorphic isomorphism. This clearly leaves at most four possibilities for the isomorphism class of X'. □

REMARK 3.2. We observe that the group G is an invariant of the homotopy class of X, and so are the curves S_i, up to complex conjugacy, and their types.

In particular any holomorphic isomorphism between Beauville surfaces X and X' induces an isomorphism between the corresponding curves S_i and S_i'. Thus the group G, the curves S_i and the types of the orbifolds S_i/G are invariants of the isomorphism class of X.

3.4. Non-homeomorphic conjugate Beauville structures on PSL(2,p)

It was proved by Bauer, Catanese and Grunewald in [5] that $\mathrm{PSL}(2,p)$ admits Beauville structure for every prime $p > 5$, a result later generalized to $\mathrm{PSL}(2,q)$ for prime powers $q > 5$ by Fuertes and Jones [26] and Garion [29] (see also [28]). In this section we will construct Beauville surfaces with group $\mathrm{PSL}(2,p)$ whose Galois orbits contain surfaces with non-isomorphic fundamental group. First of all, we should note that Beauville surfaces are in fact complex surfaces defined over $\overline{\mathbb{Q}}$. This is a consequence of a version of Belyi's Theorem for complex surfaces, in which Belyi functions are replaced by Lefschetz functions (see [38]).

Now, Catanese's rigidity results suggest that Beauville surfaces should provide a fertile source of examples of such surfaces. Indeed any Beauville surface $X = S_1 \times S_2/G$, where S_1, S_2 are curves of genera $g(S_1) \neq g(S_2)$ such that there is a $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ so that S_1^σ is not isomorphic to S_1 or $\overline{S_1}$ will be not homeomorphic to X^σ. The problem is that, as far as we know, the only examples of Beauville surfaces in which the algebraic equations of the curves S_i are explicitly given are Beauville's own examples, in which $S_1 = S_2$ is a Fermat curve $x^n + y^n + z^n = 0$ and it is easy to see that in that case $X^\sigma = X$ for every Galois element σ ([40]). Rather, the construction of Beauville surfaces with Beauville group G is usually achieved by choosing a pair of triples of generators (a_i, b_i, c_i) of G satisfying Criterion (3.3), and in general there is no way to figure out what the action of σ on these generators looks like.

To explain the importance of these examples we recall that, by Hodge's Theorem, the dimensions of the cohomology groups $H^i(X,\mathbb{C})$ of a complex projective variety X can be expressed in terms of the Hodge numbers $h^{p,q}(X) = \dim H^p(X,\Omega^q)$ which, by Serre's GAGA principle, remain invariant under Galois conjugation. It follows that the most standard topological invariants, namely the Betti numbers and the signature of a complex projective surface are Galois invariant (see for example [**66**] Th. 6.33). Nevertheless in 1964 J. P. Serre ([**57**]) gave an example of a complex projective surface possessing non-homeomorphic Galois conjugates. Several instances of this or similar phenomena have been found since then (see [**1, 2, 21, 15, 53, 8, 58, 23**]). Another important property of our examples is that, while the fundamental groups $\pi_1(X)$ and $\pi_1(X^\sigma)$ are not isomorphic, their profinite completions are. This will be a direct consequence of Grothendieck's theory of the algebraic fundamental group of algebraic varieties.

First we consider Beauville surfaces with group $\mathrm{PSL}(2,7)$ and pair of genera $(8,49)$, which turns out to be the minimum for which this phenomenon occurs. We find that there are only two of them, that they form a complete orbit under the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and that they are not homeomorphic to each other.

Then for $p > 7$ we construct Beauville surfaces with group $\mathrm{PSL}(2,p)$, whose Galois orbits contain an arbitrarily large number of pairwise non-homeomorphic Beauville surfaces.

3.4.1. The case $\mathrm{PSL}(2,7)$**.** We will deal with Beauville structures of type $((3,3,4),(7,7,7))$ in the group $\mathrm{PSL}(2,7)$. Let (a_1,b_1,c), (a_1',b_1',c), (a_2,b_2,c) and (a_2',b_2',c) be the $(3,3,4)$ triples of generators of $\mathrm{PSL}(2,7)$ in section 1.5.3 and (u,v,w) and (u^{-1},v',w^{-1}) be the $(7,7,7)$ triples introduced in Example 1.2.

Thanks to Corollary 3.1 we can introduce the following Beauville surfaces:

- X_1 defined by the pairs of triples (a_1,b_1,c) and (u,v,w);
- X_2 defined by the pairs of triples (a_2,b_2,c) and (u,v,w);

With the notation of section 1.5 these surfaces can be written as

$$X_1 = \frac{D_1 \times D}{G_1}, \qquad X_2 = \frac{D_2 \times D}{G_2}$$

where $G_1 \cong \mathrm{PSL}(2,7)$ (resp. $G_2 \cong \mathrm{PSL}(2,7)$) is a subgroup of $\mathrm{Aut}(D_1 \times D)$ (resp. a subgroup of $\mathrm{Aut}(D_2 \times D)$).

Note that the compatibility condition (3.3) in the Criterion is automatically satisfied, since the orders involved in each of the two triples are coprime. We have the following

Theorem 3.6. *For the surfaces X_1 and X_2 constructed above the following statements hold:*

(i) *they are the only Beauville surfaces with group $G = \mathrm{PSL}(2,7)$ and curves of genera 8 and 49;*
(ii) *they constitute a complete orbit for the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$;*
(iii) *they have non-isomorphic fundamental groups, hence they are not homeomorphic to each other;*
(iv) *their pair of genera $(8,49)$ is the minimum (in the lexicographic order) for which non-homeomorphic Galois conjugate Beauville surfaces exist.*

Proof. *(i)* It can be seen that any pair of triples of $\mathrm{PSL}(2,7)$ producing a Beauville surface with curves of genera 8 and 49 have to have type $(3,3,4)$ and

$(7,7,7)$ respectively (see for example [25], Theorem 13). By Corollary 3.2, when defining Beauville surfaces we can consider triples of generators up to the action of $\mathrm{I}(G;l,m,n)$. Therefore the surfaces defined by the following pairs of triples account for all the Beauville surfaces of this type:

I.	$(a_1,b_1,c\,;\,u,v,w)$,	V.	$(a_2,b_2,c\,;\,u,v,w)$,
II.	$(a_1,b_1,c\,;\,u^{-1},v',w^{-1})$,	VI.	$(a_2,b_2,c\,;\,u^{-1},v',w^{-1})$,
III.	$(a_1',b_1',c\,;\,u^{-1},v',w^{-1})$,	VII.	$(a_2',b_2',c\,;\,u^{-1},v',w^{-1})$,
IV.	$(a_1',b_1',c\,;\,u,v,w)$,	VIII.	$(a_2',b_2',c\,;\,u,v,w)$.

Note that X_1 and X_2 are defined by the pairs of triples I and V respectively.

Now I and III define the same Beauville surface. In fact by the results in section 1.5 the triples (a_1,b_1,c) and (a_1',b_1',c) are related by an element $\varphi_1\in\mathrm{Aut}(G)\setminus G$ and similarly there exists $\varphi_2\in\mathrm{Aut}(G)\setminus G$ relating (u,v,w) and (u^{-1},v',w^{-1}). Since $[\mathrm{Aut}(G):G]=2$ we know that $\varphi_2=\varphi_1\varphi$ for some inner automorphism η. Therefore both triples are related by the diagonal action of φ_1 composed with the action of $\mathrm{Id}\times\eta$ and so our claim follows from Corollary 3.2. An analogous argument shows that the Beauville surfaces defined by II and IV, by V and VII and by VI and VIII are also pairwise isomorphic.

We now claim that II defines the same surface as I (resp. VI defines the same surface as V). In order to prove it, we first note that the pairs of triples $(a_i,b_i,c\,;\,u,v,w)$ and $(a_ib_ia_i^{-1},a_i,c\,;\,u,v,w)$ for $i=1,2$ define isomorphic Beauville surfaces by Corollary 3.2. Now if we denote by ψ conjugation by $\left(\begin{smallmatrix}5&5\\2&6\end{smallmatrix}\right)\in\mathrm{PGL}(2,7)$ (resp. conjugation by $\left(\begin{smallmatrix}4&3\\4&6\end{smallmatrix}\right)\in\mathrm{PGL}(2,7)$) and by φ conjugation by $\left(\begin{smallmatrix}6&6\\5&4\end{smallmatrix}\right)\in G$ (resp. conjugation by $\left(\begin{smallmatrix}2&6\\1&0\end{smallmatrix}\right)\in G$) we see that the element ψ acting diagonally, composed with $\mathrm{Id}\times\varphi$ interchanges the triples $(a_1b_1a_1^{-1},a_1,c\,;\,u,v,w)$ and $(a_1,b_1,c\,;\,u^{-1},v',w^{-1})$ (resp. interchanges the triples $(a_2b_2a_2^{-1},a_2,c\,;\,u,v,w)$ and $(a_2,b_2,c\,;\,u^{-1},v',w^{-1})$).

(ii) The curve D^σ is isomorphic to D for each $\sigma\in\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Now, by Theorem 1.4, if $\sigma(\zeta_8)=\zeta_8^5$ the curves D_1^σ and D_2 are isomorphic and therefore, by Remark 3.2, for any such σ we have $X_1^\sigma\cong X_2$.

(iii) If $\pi_1(X_1)\cong\pi_1(X_2)$, then Theorem 3.5 would imply that D_1 would be isomorphic either to D_2 or to $\overline{D}_2$ which, by parts 3 and 5 of Theorem 1.4, is not the case.

(iv) To see the minimality of the pair $(g_1,g_2)=(8,49)$, first let us note that all Beauville surfaces with abelian Beauville group are of the form $F_n\times F_n/G_A$, where F_n is the Fermat curve of degree n, hence defined over $\mathbb{Q}$, and that the action of G_A is also Galois invariant (see Corollary 1 in [40]). It follows that all such surfaces are defined over $\mathbb{Q}$. Now the result is a consequence of Proposition 3.1. □

Theorem 3.6 also implies the following

Corollary 3.7. *The field of moduli of the Beauville surfaces X_1 and X_2 is $\mathbb{Q}(\sqrt{2})$.*

Proof. It is obvious that the inertia groups I_{X_1} and I_{D_1} (resp. I_{X_2} and I_{D_2}) coincide and the corollary follows from part (v) of Theorem 1.4. □

3.4.2. Arbitrarily large Galois orbits of Beauville surfaces with group PSL(2,p). We consider now Beauville structures of type $((2,3,n),(p,p,p))$ in the group $G = \mathrm{PSL}(2,p)$, where $n > 6$ divides either $(p-1)/2$ or $(p+1)/2$.

Clearly any pair of triples of generators of G of types $(2,3,n)$ and (p,p,p) satisfy the criterion (3.3), since the orders are coprime. Hence, with the notation of chapter 1, for any prime number $p > 5$ we can introduce the following $\phi(n)$ Beauville surfaces

- X_i defined by the pairs of triples $(\mathfrak{a}_i, \mathfrak{b}_i, \mathfrak{c}^i)$ and (u, v, w),
- X_i' defined by the pairs of triples $(\mathfrak{a}_i, \mathfrak{b}_i, \mathfrak{c}^i)$ and $(u', v', w^{\varepsilon})$,

where $i \in I := \{k : \gcd(k,n) = 1\,,\, 1 \le i < n/2\}$. Note that both of them can be written as $S_i \times S/G$, but the action of G on the product $S_i \times S$ is different in each case.

We have the following theorem.

Theorem 3.7. *Let p be a prime number $p \ge 13$ and $n > 6$ any natural number dividing either $(p-1)/2$ or $(p+1)/2$. There are exactly $\phi(n)$ isomorphism classes of Beauville surfaces with group $G = \mathrm{PSL}(2,p)$ and bitype $((2,3,n),(p,p,p))$, represented by the surfaces X_i and X_i' constructed above.*

Proof. By Proposition 3.2, when defining Beauville surfaces we can consider triples of generators up to the action of $\mathrm{I}(G; l_i, m_i, n_i)$. Therefore the surfaces defined by the following pairs of triples

$$\begin{aligned} t_1(i) &= (\mathfrak{a}_i, \mathfrak{b}_i, \mathfrak{c}^i\,;\, u, v, w)\,, & t_2(i) &= (\mathfrak{a}_i, \mathfrak{b}_i, \mathfrak{c}^i\,;\, u', v', w^{\varepsilon})\,,\\ t_1'(i) &= (\mathfrak{a}_i', \mathfrak{b}_i', \mathfrak{c}^i\,;\, u', v', w^{\varepsilon}), & t_2'(i) &= (\mathfrak{a}_i', \mathfrak{b}_i', \mathfrak{c}^i\,;\, u, v, w), \end{aligned}$$

for $1 \le i < n/2$ with $\gcd(i,n) = 1$ account for all the Beauville surfaces of this type. Note furthermore that each X_i and X_i' are defined by the pairs of triples $t_1(i)$ and $t_2(i)$ respectively.

Now, the pairs of triples $t_1(i)$ and $t_1'(i)$ (resp. $t_2(i)$ and $t_2'(i)$) define the same Beauville surface. In fact, by the two lemmas above any element of $\mathrm{Aut}(G) \setminus G$ sends the triple $(\mathfrak{a}_i, \mathfrak{b}_i, \mathfrak{c}^i)$ to a triple $\mathrm{I}(G; 2, 3, n)-$equivalent to $(\mathfrak{a}_i', \mathfrak{b}_i', \mathfrak{c}^i)$, and (u, v, w) to a triple $\mathrm{I}(G; p, p, p)-$equivalent to $(u', v', w^{\varepsilon})$, and the claim follows from Corollary 3.2.

However, for the same reason $t_1(i)$ and $t_2(i)$ define non-isomorphic Beauville surfaces since, by Corollary 3.2, this happens if and only if there exists $\psi \in \mathrm{Aut}(G)$ such that

$$\begin{aligned} \psi(\mathfrak{a}_i, \mathfrak{b}_i, \mathfrak{c}^i) &\equiv (\mathfrak{a}_i, \mathfrak{b}_i, \mathfrak{c}^i) \bmod \mathrm{I}(G; 2, 3, n)\,,\\ \psi(u', v', w^{\varepsilon}) &\equiv (u, v, w) \bmod \mathrm{I}(G; p, p, p)\,, \end{aligned}$$

simultaneously. Now, the first relation may occur only if $\psi \in G$, and the second one only if $\psi \notin G$.

On the other hand, if $i \ne j$, Corollary 3.3 implies that the surfaces defined by $t_1(i)$ and $t_2(i)$ and the ones defined by $t_1(j)$ and $t_2(j)$ cannot be isomorphic, since the Riemann surfaces of type $(2,3,n)$ involved in the construction of the first ones are not isomorphic to the ones appearing in the second ones.

Finally, the condition $p \ge 13$ follows from the fact that for prime numbers p with $5 < p < 13$ there are no natural numbers $n > 6$ dividing either $(p-1)/2$ or $(p+1)/2$. □

Now, the Beauville surface X_1 can be written as $E_1 \times E/G$. Since, by Theorem 1.2, for any Galois element σ such that $\sigma(\zeta_n) \ne \zeta_n^{\pm 1}$ we have $E_1^{\sigma} \ncong E_1$ and

$E_1^\sigma \not\cong \overline{E_1}$, we have at least $\phi(n)/2$ non-homeomorphic conjugate Beauville surfaces

$$X_1^{\sigma_i} = \frac{E_i \times E}{\mathrm{PSL}(2,p)},$$

where σ_i are Galois elements satisfying $\sigma_i(\zeta_n) = \zeta_n^j$ with $ij \equiv 1 \mod n$.

As a consequence we have the following.

Theorem 3.8. *For each prime number $p > 7$ and each integer $n > 6$ dividing either $(p-1)/2$ or $(p+1)/2$ there exist a Beauville surface $X_1 = E_1 \times E/G$ with $G = \mathrm{PSL}(2,p)$ such that the following statements hold:*

(i) *E_1 and E are $G-$coverings of type $(2,3,n)$ and (p,p,p) respectively;*
(ii) *the orbit of X_1 under the action of the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ contains at least $\phi(n)/2$ surfaces which are pairwise non-isomorphic.*
(iii) *in fact, these $\phi(n)/2$ surfaces have pairwise non-isomorphic fundamental groups, hence they are not homeomorphic to each other.*

We can prove the same theorem for the surface X_1', thus we have two sets of $\phi(n)/2$ Beauville surfaces, and each of them consists of Galois conjugate surfaces which are pairwise non-homeomorphic. It is not clear, however, whether these two sets form a complete orbit or two separate orbits under the action of the absolute Galois group.

With regard to this issue it should be said that, after a conversation on this matter with G. A. Jones, he soon realised that replacing $\mathrm{PSL}(2,p)$ by $\mathrm{PGL}(2,p)$ one could construct complete orbits of the absolute Galois group of explicit unbounded size, consisting of Beauville surfaces with mutually non-isomorphic fundamental groups (see [41]). In this case one is able to compute explicitly the complete orbit thanks to the fact that the groups $\mathrm{PGL}(2,p)$ are complete, i.e. all their automorphisms are inner. The methods used to compute the triples defining the surfaces are similar to those used in section 1.5. The main theorem in [41], whose results we include here only for completeness, states that for each prime $p \equiv 19 \mod 24$ and each pair of divisors $k, l > 10$ of $p-1$ and $p+1$ such that $(p-1)/k$ and $(p+1)/l$ are odd, there is an orbit of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ consisting of $\phi(m)/4$ Beauville surfaces with Beauville group $\mathrm{PGL}(2,p)$ which have mutually non-isomorphic fundamental groups, where $m = \mathrm{lcm}(k,l)$.

With respect to the question of determining the fields of definition of Beauville surfaces, raised by Bauer, Catanese and Grunewald (see [6]), the above theorem shows that minimal fields of definition of Beauville surfaces can have arbitrarily large degree over the field of rational numbers.

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