GAUS AG Buildings - WS22/23

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1 Introduction

Buildings are large simplicial complexes with many symmetries. Requiring that they are large enough, and that their isomorphisms group G acts in a sufficiently transitive way, makes them into rather rigid objects. Buildings were introduced by Tits in order to study the structure of (and classify) reductive algebraic groups. Buildings are constructed by gluing *apartments* - spheres or Euclidean spaces, subdivided according to the action of a(n affine) Weyl group - along *convex* subcomplexes. Typically, apartments correspond to maximal tori (flats) in the group. In the spherical case, the simplices in one apartment correspond to the parabolic subgroups containing the associated maximal torus. In the Euclidean case, the building can be viewed as a sort of skeleton for the analytification of the flag variety G/B. Some finite(ly generated) subcomplexes of the building play a prominent book-keeping role in some problems of algebraic geometry, such as T-equivariant principal bundles on toric varieties, and moduli of hyperplane arrangements. Buildings mimic the structure at infinity of locally symmetric spaces; they have found applications in the study of toroidal compactifications of the latter, thus relating to the subject of Shimura varieties.

2 Schedule & plan

Time: Tuesday 2-3:30 pm, October 18th - February
Room: Raum 110, Robert-Mayer-Str. 6-8, Frankfurt + Zoom
   `https://uni-frankfurt.zoom.us/j/62173091959?pwd=eVFsaE5ndm1hVGdXQk5XTHZKWHBRZz09`

We will schedule weekly meetings consisting of a one-hour talk plus some time for Q&A and further discussion. The participants are encouraged to read the material in advance of the talk. The first seven talks are introductory and we plan to follow them rather closely. The remaining talks are meant to cover an array of applications and deeper results; participants' suggestions are more than welcome. The talks will take place in Frankfurt as long as possible, but we will live-stream them as long as there is a demand.

2.1 Part I: basic theory

18.10.2022: Finite reflection groups and their associated complexes [Kevin Kühn] Define Coxeter groups as groups generated by reflections along a finite hyperplane arrangement in a Euclidean vector space [AB08, Chapters 1]. Mention the classification [AB08, §1.3] and draw some low-rank examples, using them to introduce some jargon (roots, chambers, panels, galleries, simplicial and chamber complexes...). Overview the main properties [AB08, §1.5.4 and §1.6]. C.f. [Bro91, Appendix A].

25.10: Coxeter groups and complexes in general [Arne Kuhrs] Define Coxeter groups as in [AB08, §2.1] - in this talk, leave out as many group-theoretic subtleties as possible; some of them will be reviewed in Talk 4. Time permitting, explain the canonical representation and its dual [AB08, §2.5], possibly in the example of the infinite dihedral group [AB08, §2.2.2]. Define the Coxeter complex [AB08, §3.1]. Introduce foldings (and explain their construction), with the aim of overviewing [AB08, Theorem 3.65].

01.11: Thick buildings - axioms and first properties [Pedro Souza] Introduce the first set of axioms for thick buildings [Bro91, Lecture 1]. Draw a tree (and other examples if possible) and show that it satisfies the axioms. Give an overview of retractions [AB08, §4.4], the product structure [AB08, §3.6 and §4.9], and convexity [AB08, §3.6 and §4.11]. Use these notions to sketch the first consequences of the axioms [Bro91, Lecture 1].

08.11: Buildings as W-metric spaces [Luca Battistella] Explain Tits' solution to the word problem for Coxeter groups, and its relation to galleries, in order to motivate the following: definition of buildings as W-metric spaces $(\mathcal{C}, \delta_W)$ [Bro91, Lecture 2]. Introduce residues and projections (from the metric standpoint) in order to recover the simplicial complex associated to the chamber set $\mathcal{C}$. Show that apartments exist (isometry extension lemma), and that the metric and the simplicial definitions are equivalent.

22.11: Group actions and BN pairs [Jiaming Chen] Introduce the various notions of transitivity for a group action on a building. Explain how a transitive action gives rise to a BN-pair, and the relation with the theory of algebraic groups. If time permits, talk about the Bruhat decomposition. Follow [Bro91, Lectures 3-4] and [AB08, §6.1-2].

29.11: Examples of buildings [Alejandro Vargas] Describe the building of $PGL_n(K)$ when K has a trivial (in terms of flags) or discrete (in terms of lattices or norms) valuation [AB08, §§6.5 and 6.9], and the spherical bulding of Sp_{2n} [AB08, §6.6]. See also [BT87]. Optional: explain the permutation-valued distance between flags (sometimes this is called Jordan–Hölder permutation).

06.12: Euclidean buildings [Andreas Gross] Explain the CAT(0) property and the spherical building at infinity, following [Bro91, Lectures 4.4 and 5] or [AB08, Chapter 11].

2.2 Part II: applications

13.12: Bordifications of reductive groups [Martin Ulirsch] The theory of toric varieties provides us with a combinatorial dictionary to construct and describe bordifications of algebraic tori T. Mumford's insight was that such bordifications can be extended to solvable groups $B > T$, and that, if the chosen fan is Weyl-invariant, these bordifications can be glued together as B ranges through the Borel subgroups of a fixed semisimple group G (parametrised by the spherical building). Here we expand on [KKMSD73, Ch.IV,§2].

20.12: Toric principal bundles [Felix Röhrle] Torus-equivariant principal G-bundles on toric varieties correspond to piecewise-linear maps from the fan to the (cone over) the spherical building of G, see [KM18]. See also [Kly89] for the case of $G = GL$.

10.01: NO seminar

17.01: The Bruhat-Tits construction and the relation with reductive groups [Annette Werner] This talk discusses the construction by Bruhat and Tits, which associates a Euclidean building to a reductive group over a suitable valued ground field.

24.01: Normed fibre functors [Paul Ziegler, TU Munich] [Zie22]

31.01: Langton's theorem [Johannes Horn] Existence of limits for one-parameter families of semistable torsion-free sheaves [Lan75], or its generalisation to Higgs bundles [Nit91].

Thursday 09.02: Algebraic loop groups and their flag varieties [Timo Richarz, TU Darmstadt] Loop groups and their associated flag varieties naturally occur in geometric approaches to the Langlands program. Prominent examples include so-called affine Grassmannians. In this talk I give an introduction to the topic focussing on algebro-geometric aspects and relations with Bruhat-Tits theory.

Other topics:

Compactifications and Berkovich geometry Define the compactification of the affine building of PGL in terms of seminorms (resp. free submodules of smaller rank). Explain how it includes into (and is retracted upon by) $\mathbb{P}^{\mathrm{an}}$, see for instance [Wer04, RTW15].

Classification Give an overview of Tits' rigidity and extension theorems [AB08, §5.9-10], the Moufang property, and the classification of spherical buildings of rank at least three [AB08, Chapters 7 and 9].

Tropical convexity Compare the various notions of convexity in the affine building PGL. Show that $+/\cap$-convexity corresponds to tropical convexity with the min / max convention. Explain the relationship between membranes, valuated matroids, and tropical linear spaces. C.f. [DS04].

Moduli of hyperplane arrangements Limits of configurations of hyperplanes in $\mathbb{P}^n$ can be computed using the affine building of PGL, see [Kap93] for the case of points in $\mathbb{P}^1$, and [KT06] for the general case.

References

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[Kly89]   A. A. Klyachko. Toric bundles and problems in linear algebra. *Funktsional. Anal. i Prilozhen.*, 23(2):63–64, 1989.

[KM18]   Kiumars Kaveh and Christopher Manon. Toric principal bundles, piecewise linear maps and Tits buildings. *arXiv e-prints*, page arXiv:1806.05613, June 2018.

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